

10-1-1992

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Chris Bernhardt

Fairfield University, cbernhardt@fairfield.edu

E. Coven

M. Misiurewicz

Irene Mulvey

Fairfield University, mulvey@fairfield.edu

First published in Transactions of the American Mathematical Society in 333 (2), published by the American Mathematical Society.

Peer Reviewed

Repository Citation

Bernhardt, Chris; Coven, E.; Misiurewicz, M.; and Mulvey, Irene, "Comparing Periodic Orbits of Maps of the Interval" (1992). *Math & Computer Science Faculty Publications*. Paper 4.
<http://digitalcommons.fairfield.edu/mathandcomputerscience-facultypubs/4>

Published Citation

C. Bernhardt, E. Coven, M. Misiurewicz, and I. Mulvey. *Comparing Periodic Orbits of Maps of the Interval*, Transactions of the American Mathematical Society. 333(2), 701-707.

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COMPARING PERIODIC ORBITS OF MAPS OF THE INTERVAL

C. BERNHARDT, E. COVEN, M. MISIUREWICZ, AND I. MULVEY

ABSTRACT. Let π and θ be cyclic permutations of finite ordered sets. We say that π forces θ if every continuous map of the interval which has a representative of π also has one of θ . We give a geometric version of Jungreis' combinatorial algorithm for deciding in certain cases whether π forces θ .

INTRODUCTION

This paper is concerned with the *forcing relation* between finite cyclic permutations. Let π be a cyclic permutation of $\{1, \dots, k\}$ and let $f: I \rightarrow I$ be a continuous map of a compact interval. A *representative* of π in f is a subset $\{x_1 < \dots < x_k\}$ of I such that $f(x_i) = x_{\pi(i)}$ ($i = 1, \dots, k$). We say that π forces θ if every continuous map of a compact interval which has a representative of π also has a representative of θ . The study of the forcing relation is a natural outgrowth of the proof in [BGM] of Sarkovskii's Theorem.

The formal study of the forcing relation was begun by S. Baldwin [Ba], who showed that forcing is a partial order and gave an algorithm for deciding whether π forces θ . Baldwin's algorithm checks, in terms of closed walks through directed graphs with oriented edges (oriented Markov graphs), whether θ has a representative in the *canonical π -linear map* L_π , defined on $[1, k]$ by $L_\pi = \pi$ on $\{1, \dots, k\}$ and L_π is linear on $[1, 2], \dots, [k-1, k]$. I. Jungreis [J, Theorem 5.1] gave a faster algorithm, in terms of itineraries, for deciding whether π forces θ in certain cases.

In this paper, we present an effective algorithm for deciding whether π forces θ in the cases considered by Jungreis. Our algorithm is in some sense the same as Jungreis'. However, our implementation is in terms of rational numbers instead of itineraries, and our proof is geometric instead of combinatorial, avoiding the "intricate combinatorial theorem" of [J, §6]. The main idea of our proof is to compare canonical representatives of π and θ in maps we call *horseshoe maps*.

Throughout this paper, the lower case Greek letters π, θ, η will denote cyclic permutations, and $\pi = (a_1 a_2 \dots a_n)$ means that $\pi(a_1) = a_2, \dots, \pi(a_{n-1}) = a_n, \pi(a_n) = a_1$. The upper case Latin letters P, Q, R will denote representatives of cyclic permutations, i.e., periodic orbits. The upper case Latin letter H will denote a horseshoe map (defined in the next section).

Received by the editors July 13, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 58F20, 58F08, 58H20.

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0002-9947/92 \$1.00 + \$.25 per page

1. BACKGROUND

Let $n \geq 2$. The horseshoe map $H : [0, 1] \rightarrow [0, 1]$ of type n^+ is defined as follows: for $x \in [i/n, (i + 1)/n]$ ($i = 0, \dots, n - 1$),

$$H(x) = \begin{cases} n(x - i/n) & \text{if } i \text{ is even,} \\ 1 - n(x - i/n) & \text{if } i \text{ is odd.} \end{cases}$$

H is continuous, maps each of the n laps

$$I_1 = [0, 1/n], I_2 = [1/n, 2/n], \dots, I_n = [(n - 1)/n, 1]$$

linearly onto $[0, 1]$, and is increasing on the first lap. The horseshoe map of type n^- is defined analogously. It is decreasing on the first lap. In either case, we write $l(H) = n$.

Let $n \geq 0$. We say that π has type n^+ (resp. n^-) if the canonical π -linear map L_π has n laps, and is increasing (resp. decreasing) on the first lap. In either case, we write $l(\pi) = n$. Note that the trivial permutation (1) has type 0^+ and type 0^- . For any π and any H , the set of representatives of π in H is finite. It is nonempty if and only if π and H have the same type or $l(\pi) < l(H)$ [MN].

An important tool in the early study of forcing was the observation by Baldwin [Ba, Lemma 3.1 and Theorem 3.3] (see also [Be1, Lemmas 1.7 and 1.8]) that if θ has a representative in the canonical π -linear map L_π , then π forces θ . We shall use a variant of this result involving horseshoe maps.

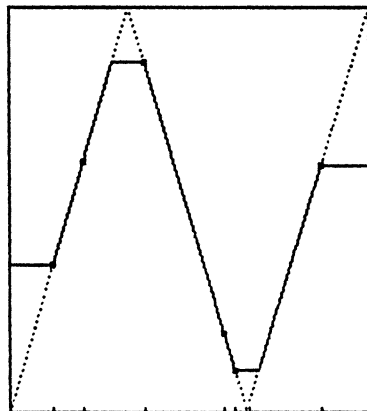
Suppose P is a periodic orbit of H . For $i = 0, \dots, n$ ($n = l(H)$), define $d_i = \min\{|p - i/n| : p \in P\}$ and let \bar{p}_i be the unique point in P such that $|\bar{p}_i - i/n| = d_i$. We say that P fits H if $n = 2$ or if $n \geq 3$ and $d_i + d_{i+1} \leq 1/n$ ($i = 1, \dots, n - 2$). In either case, the P -truncation H_p of H , defined by

$$H_p(x) = \begin{cases} H(\bar{p}_i) & \text{if } |x - i/n| \leq d_i, \quad (i = 0, 1, \dots, n) \\ H(x) & \text{otherwise,} \end{cases}$$

is well defined. The figure below shows the graph of H_p , where H is the horseshoe map of type 3^+ and

$$P = \{46/365, 76/365, 138/365, 218/365, 228/365, 316/365\}$$

is a representative of (136425) in H .



Remark. If P does not fit H , then H_p may not be well defined. For example, consider the periodic orbit $\{4/63, 16/63, 62/63\}$ of the horseshoe map of type $4+$.

Lemma 1. *Let P be a representative of π in H such that H_p is well defined. If θ has a representative in H_p , then π forces θ .*

Proof. For $x \in [0, 1]$, let x' (resp. x'') be the right (resp. left) endpoint of the largest closed interval containing x on which H_p is constant. If $P = \{x_1 < \dots < x_k\}$, set $x_0 = 0$ and $x_{k+1} = 1$. Then every periodic orbit of H_p lies in $\bigcup_{i=0}^k [x'_i, x''_{i+1}]$. Therefore by [MN, Theorem 1.21], π forces θ . \square

We say that π fits H if the set of representatives of π in H is nonempty and every representative of π in H fits H .

We leave the proof of the following lemma to the reader.

Lemma 2. *π fits H if and only if one of the following conditions holds:*

- (1) π and H have the same type.
- (2) $l(\pi) = l(H) - 1$.
- (3) H has type $n \pm$ and π has type $(n - 2) \mp$.

Suppose that P and Q are periodic orbits of H such that H_Q is well defined. We say that P is tighter than Q if $H_Q|P = H|P$, i.e., H and H_Q agree on P .

Lemma 3. *Suppose that P and Q are periodic orbits of H such that H_Q is well defined. Then the following statements are equivalent:*

- (1) P is tighter than Q .
- (2) $d_i(P) \geq d_i(Q)$ ($i = 0, \dots, l(H)$).

If, in addition, P has at least three members, then the following statement is equivalent to the others:

- (3) $d_i(P) \geq d_i(Q)$ ($i = 1, \dots, l(H) - 1$).

Proof. The equivalence of (1) and (2) follows from the definitions. We show that (3) implies (1) if P has at least three members.

Let $n = l(H)$ and let \bar{H}_Q be the map defined analogously to H_Q , except that we do not truncate at 0 or 1, i.e., we ignore d_0 and d_n . Then P is a periodic orbit of \bar{H}_Q . Since $\bar{H}_Q = H_Q$ on $[1/n - d_1, (n - 1)/n + d_{n-1}]$ and this interval is mapped to itself by H_Q , it suffices to show that $P \subseteq [1/n - d_1, (n - 1)/n + d_{n-1}]$.

The complement of $[1/n - d_1, (n - 1)/n + d_{n-1}]$ consists of two intervals, K_1 and K_2 . H is linear on these two intervals, either $H(K_1) \cap K_1 = \emptyset$ or $H(K_1) \cap K_2 = \emptyset$, and either $H(K_2) \cap K_1 = \emptyset$ or $H(K_2) \cap K_2 = \emptyset$. Therefore any periodic orbit of H which lies in $K_1 \cup K_2$ can have only one or two members. \square

In order to use Lemmas 1 and 3 effectively, we need a method of constructing representatives of a cyclic permutation in a horseshoe map. The tool we shall use is itineraries.

An H -itinerary (itinerary for short when H is clear) of a point $x \in [0, 1]$ is a sequence $\tau = (\tau(j))_{j=0}^\infty$ such that $H^j(x) \in I_{\tau(j)}$ ($j = 0, 1, \dots$). The correspondence between points and itineraries is one-to-one, except for those points whose orbit meets the set of turning points of H . Such points have two itineraries. Since H maps the turning points to 0 or 1, this cannot happen for

periodic points. Thus there is a one-to-one correspondence between periodic points and periodic itineraries. This correspondence preserves (least) periods.

Itineraries are ordered in the following way. Define $\varepsilon_k = +1$ (resp. -1) if H is increasing (resp. decreasing) on I_k , $\zeta(0) = +1$, and $\zeta(k) = \varepsilon_{\tau(0)} \cdots \varepsilon_{\tau(k-1)}$ ($k \geq 1$). If $\tau \neq \sigma$ are itineraries, let k be the least integer such that $\tau(k) \neq \sigma(k)$. Then $\tau < \sigma$ if $\zeta(k) = +1$ and $\tau(k) < \sigma(k)$, or if $\zeta(k) = -1$ and $\tau(k) > \sigma(k)$. In the other two cases, $\tau > \sigma$.

Lemma 4 (cf. [MT, Lemma 3.1]). *Let x and y be points in $[0, 1]$ having unique itineraries τ and σ . Then $x < y$ if and only if $\tau < \sigma$.*

Theorem 5. *Let $l(H) = n$ and let τ be a sequence with $\tau(i) \in \{1, \dots, n\}$ ($i = 0, 1, \dots$). Then the unique point in $[0, 1]$ whose itinerary is τ is $\sum_{j=0}^{\infty} \alpha_j/n^{j+1}$, where $\alpha_j = \tau(j) - 1$ if $\zeta(j) = +1$, $\alpha_j = n - \tau(j)$ if $\zeta(j) = -1$.*

Proof. Using induction on m , it is straightforward to show that for $m \geq 0$, the left endpoint of $I_{\tau(0)} \cap H^{-1}(I_{\tau(1)}) \cap \cdots \cap H^{-m}(I_{\tau(m)})$, which has length $1/n^{m+1}$, is $\sum_{j=0}^m \alpha_j/n^{j+1}$. \square

We say that π is a *doubling* of η if for some even k , π is defined on $\{1, \dots, k\}$, η is defined on $\{1, \dots, k/2\}$, and $\pi\{2i - 1, 2i\} = \{2\eta(i) - 1, 2\eta(i)\}$ ($i = 1, \dots, k/2$). In this case, we write $\eta = \pi/2$.

Lemma 6 [Be2, Theorem 1.12].

- (1) π forces $\pi/2$.
- (2) If π forces $\theta \neq \pi$, then $\pi/2$ forces θ .

2. THE ALGORITHM

In the propositions below, π is defined on $\{1, \dots, k\}$, π fits H , and $n = l(H)$. Label the laps of L_π from left to right as follows:

- (1) J_1, \dots, J_n if H and π have the same type,
- (2) J_1, \dots, J_{n-1} if H has type $n \pm$ and π has type $(n - 1) \pm$,
- (3) J_2, \dots, J_n if H has type $n \pm$ and π has type $(n - 1) \mp$,
- (4) J_2, \dots, J_{n-1} if H has type $n \pm$ and π has type $(n - 2) \mp$.

In cases (2), (3), and (4), we allow L_π to have “phantom” laps J_1 and/or J_n . In this way, L_π is increasing or decreasing on J_m according as H is increasing or decreasing on I_m ($m = 1, \dots, n$).

We define below an itinerary τ which is the largest L_π -itinerary of 1, even allowing “phantom” itineraries, where the laps of L_π are labelled as above.

If $\pi = (1)$, then H has type $2 \pm$. In this case, let $\tau = 111 \cdots$ if H has type $2-$, and $\tau = 222 \cdots$ if H has type $2+$.

If $\pi \neq (1)$, we define τ inductively as follows. Let $\tau(0) = 1$ if (1) or (2) hold, $\tau(0) = 2$ if (3) or (4) hold. Assume that $\tau(0), \dots, \tau(j - 1)$ have been defined. If $L_\pi^j(1) \in J_i$ only, let $\tau(j) = i$. If $L_\pi^j(1) \in J_i \cap J_{i+1}$, let $\tau(j) = i$ if $\zeta(j) = -1$, $\tau(j) = i + 1$ if $\zeta(j) = +1$. (Here $1 \in J_1 \cap J_2$ in cases (3) and (4), and $k \in J_{n-1} \cap J_n$ in cases (2) and (4).)

Let r be the unique point in $[0, 1]$ whose H -itinerary is τ , and let $R(\pi)$ be the orbit of r . Since τ is an L_π -itinerary of 1, $r = \min R(\pi)$. In the sequel, ζ corresponds to the itinerary τ defined above.

Proposition 7 below is straightforward.

Proposition 7. *If L_π has a turning point at $L_\pi^j(1)$ which is a relative maximum (resp. minimum), then $\zeta(j + 1) = -1$ (resp. $+1$).*

(Here 1 is a turning point of L_π in cases (3) and (4), and k is a turning point in cases (2) and (4).)

Proposition 8. *$R(\pi)$ is a representative of π or $\pi/2$.*

Proof. The result is true for $\pi = (1)$. If $\pi = (12)$, the result is true since $\tau = 111\dots$ if H has type $2-$ and $\tau = 222\dots$ if H has type $2+$ and $3+$. So assume that $\pi \neq (1)$ or (12) , and hence that $k \geq 3$.

We show that the period of $R(\pi)$ divides k . To do this, it suffices to show that $\zeta(k) = +1$, for then τ and hence r have periods which divide k . Since $L_\pi^k(1) = 1$, either L_π has a turning point at $L_\pi^{k-1}(1)$ which is a relative minimum, or $L_\pi^{k-1}(1) = k \in J_n$ and L_π is decreasing on J_n . In the first case, $\zeta(k) = +1$ by Proposition 7. In the second case, $\varepsilon_{\tau(k-1)} = -1$ and L_π has a turning point at $L_\pi^{k-2}(1)$ which is a relative maximum. Therefore by Proposition 7, $\zeta(k - 1) = -1$ and so $\zeta(k) = +1$.

Since $k \geq 3$, $R(\pi)$ cannot lie on a single lap of H , and therefore the period of $R(\pi)$ is greater than one. This period is k/s , where $1 \leq s < k$. Now L_π permutes the blocks $\{1, \dots, s\}, \{s + 1, \dots, 2s\}, \dots, \{k - s + 1, \dots, k\}$, and is monotone on each block. Therefore $s = 1$ or 2 .

The map $\varphi: \{1, \dots, k\} \rightarrow R(\pi)$, defined by $\varphi(i) = H^J(r)$, where $i = L_\pi^j(1)$, preserves the order relation \leq . Thus $R(\pi)$ is a representative of π or $\pi/2$ according as $s = 1$ or 2 . \square

Proposition 9. *$R(\pi)$ is a representative of $\pi/2$ or π according as π is a doubling or not.*

Proof. If π is not a doubling, then $\pi/2$ is not defined, and so by Proposition 8, $R(\pi)$ is a representative of π . Suppose then that π is a doubling. Let $L_\pi^j(1) \in J_1 \cap J_{i+1}$, where $0 \leq j \leq k/2$. If $L_\pi^{j+k/2}(1) < L_\pi^j(1)$, then $\zeta(j) = -1$ and $\zeta(j + k/2) = +1$, while if $L_\pi^{j+k/2}(1) > L_\pi^j(1)$, then $\zeta(j) = +1$ and $\zeta(j + k/2) = -1$. In either case, $\tau(j + k/2) = \tau(j)$. Therefore τ and hence r have period $k/2$. Thus $R(\pi)$ is not a representative of π , and so by Proposition 8 it must be a representative of $\pi/2$. \square

Proposition 10. *If π is a doubling, then $\zeta(k/2) = -1$.*

Proof. Setting $j = k/2$ in the proof of Proposition 9, we get $L_\pi^k(1) < L_\pi^{k/2}(1)$, and hence $\zeta(k/2) = -1$. \square

Proposition 11. *$R(\pi)$ fits H .*

Proof. We need only consider the case that π is a doubling. We may assume that $n \geq 3$, for if $n = 2$ there is nothing to prove.

By Proposition 10, $\zeta(k/2) = -1$. Then by [MN, Proposition 9.8(1)], the orbit P of the largest fixed point of H^k less than r (such exists) is a representative of a cycle η which is a doubling of $\pi/2$ and such that each point of $R(\pi)$ is surrounded by two points of P , in the sense that its nearest neighbors in $R(\pi) \cup P$ both are in P . Furthermore, if $R(\pi) = \{r_1 < \dots < r_{k/2}\}$, then η is the unique doubling of $\pi/2$ such that $\eta(2i) - \eta(2i - 1)$ has the same sign as the slope of H at r_i ($i = 1, \dots, k/2$). Since $\varepsilon_{\tau(j)} = +1$ or -1

according as $L_\pi^{j+k/2}(1) - L_\pi^j(1)$ and $L_\pi^{j+k/2+1}(1) - L_\pi^{j+1}(1)$ have the same sign or opposite signs ($j = 0, \dots, k/2 - 1$), π has this last property, and so P is a representative of π .

If $P = \{p_1 < \dots < p_k\}$, call the points p_{2m-1} and p_{2m} ($m = 1, \dots, k/2$) *partners*. If $d_i(P)$ is attained at $\bar{p}_i \in P$, let \tilde{p}_i be the partner of \bar{p}_i , and let $\tilde{d}_i = |\bar{p}_i - i/n|$. Since P fits H , $d_i + d_{i+1} \leq 1/n$ ($i = 1, \dots, n - 2$). Using the fact that H maps partners to partners, it is straightforward to show that $\tilde{d}_i + \tilde{d}_{i+1} \leq 1/n$ ($i = 1, \dots, n - 2$) as well. Since the point of $R(\pi)$ at which $d_i(R(\pi))$ is attained lies between \bar{p}_i and \tilde{p}_i ($i = 0, \dots, n$), we have $d_i(R(\pi)) + d_{i+1}(R(\pi)) \leq 1/n$ ($i = 1, \dots, n - 2$). \square

Remark. If π is a doubling which fits H , then although $R(\pi)$ fits H , some other representative of $\pi/2$ may not fit H . For example, $\pi = (136245)$, of type $3+$, fits the horseshoe map H of type $4+$, but $\pi/2 = (123)$ has a representative which does not fit H . See the remark before Lemma 1.

Proposition 12. $R(\pi)$ is tighter than every representative of π in H .

Proof. The result is trivial if $\pi = (1)$, so we may assume that $k \geq 2$. By looking at the points $(i, L_\pi(i))$ ($i = 1, \dots, k$) in the graph of L_π and the points $(p, H(p))$ ($p \in P$) in the graph of H , it is easy to see that for every representative P of π in H , and for $P = R(\pi)$ as well if π is a doubling, $d_i(P)$ is attained at $H^j(p)$, where $p = \min P$ and j is the unique integer, $0 \leq j \leq k - 1$, such that $L_\pi^j(1) \in J_i \cap J_{i+1}$ ($i = 1, \dots, n - 1$). Therefore to show that $d_i(R(\pi)) \geq d_i(P)$, it suffices to show that $H^{j+1}(p) \geq H^{j+1}(r)$ (resp. $H^{j+1}(p) \leq H^{j+1}(r)$) if L_π has a relative maximum (resp. minimum) at $L_\pi^j(1)$.

The itinerary σ of p is also an L_π -itinerary of 1. From the construction of τ and the definition of ordering of itineraries it follows that at each step of the construction of τ we make the choice that makes τ maximal. Hence $\sigma \leq \tau$ and so $p \leq r$. By Proposition 7, if L_π has a relative maximum at $L_\pi^j(1)$, then $H^{j+1}(p) \geq H^{j+1}(r)$, and if L_π has a relative minimum at $L_\pi^j(1)$, then $H^{j+1}(p) \leq H^{j+1}(r)$. Therefore $d_i(P) \leq d_i(R(\pi))$ ($i = 1, \dots, n - 1$), and $R(\pi)$ is tighter than P by Lemma 3. \square

Remark. It can be shown that if π is a doubling and $l(\pi) \geq 3$, then π has no tightest representative in the horseshoe map of the same type as π . The simplest example is $\pi = (136245)$.

Theorem 13 (cf. [J, Theorem 5.1]). *Let π and θ fit H . Then π forces θ if and only if $R(\theta)$ is tighter than $R(\pi)$.*

Proof. By Proposition 11, $R(\pi)$ fits H , and so $H_{R(\pi)}$ is well defined.

Assume that π forces θ . Suppose first that π is not a doubling. Then $R(\pi)$ is a representative of π in H , and therefore θ has a representative Q in $H_{R(\pi)}$. Then Q is a representative of θ in H and Q is tighter than $R(\pi)$. But $R(\theta)$ is tighter than Q by Proposition 12, so $R(\theta)$ is tighter than $R(\pi)$. Suppose now that π is a doubling. We may assume that $\theta \neq \pi$. By Lemma 6, $\pi/2$ forces θ . By Proposition 9, $R(\pi)$ is a representative of $\pi/2$ in H , and the argument above shows that $R(\theta)$ is tighter than $R(\pi)$.

Conversely, assume that $R(\theta)$ is tighter than $R(\pi)$. If θ is not a doubling, then $R(\pi)$ and $R(\theta)$ are representatives of π ($\pi/2$ if π is a doubling) and θ

in $H_{R(\pi)}$, and it follows from Lemma 1 that π ($\pi/2$ if π is a doubling) forces θ . In the parenthetical case too, π forces θ , because π forces $\pi/2$. Suppose then that θ is a doubling. By Proposition 10, $\zeta'(k'/2) = -1$ (here the primes refer to θ), so as in the proof of Proposition 11, there is a representative of θ in $H_{R(\pi)}$. As above, π forces θ . \square

If τ is periodic, then the terms of the series in Theorem 5 can be grouped to give a geometric series. Thus $R(\pi)$ consists of rational numbers which are computable in finitely many steps, as does the set of numbers $\{d_i(R(\pi))\}$. Therefore

If H is the horseshoe map of the same type as π , then Theorem 5 and 13 together give an effective algorithm for deciding whether π forces θ in each of the following cases:

- (1) π and θ have the same type,
- (2) $l(\theta) = l(\pi) - 1$,
- (3) π has type $n \pm$ and θ has type $(n - 2) \mp$.

Remark. If π is a doubling, we can sometimes decide whether π forces θ even if θ does not fit the horseshoe map of the same type as π . Formally, define $\pi_0 = \pi$, and if π_i is a doubling, define $\pi_{i+1} = \pi_i/2$: if not, π_i is not defined. Let m be the smallest integer such that π_m is not a doubling. To decide whether π forces θ , first check whether $\theta = \pi_i$ for some $i \leq m$. If so, then π forces θ by Lemma 6; if not, then again by Lemma 6, π forces θ if and only if π_m forces θ . The algorithm may apply to π_m and θ . For example, (1 7 11 4 5 10 2 8 12 3 6 9), of type $8+$, forces (13425) but not (1234), both of type $2+$.

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(C. Bernhardt and I. Mulvey) DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FAIRFIELD UNIVERSITY, FAIRFIELD, CONNECTICUT 06430
E-mail address: cbernhard@fair1.bitnet
E-mail address: itmulvey@fair1.bitnet

(E. Coven) DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT 06457
E-mail address: coven@jordan.math.wesleyan.edu

(M. Misiurewicz) INSTYTUT MATEMATYKI, UNIwersytet warszawski, 00-901 warsaw, POLAND