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Discriminating groups

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Abstract. A group G is termed *discriminating* if every group separated by G is discriminated by G. In this paper we answer several questions concerning discrimination which arose from [2]. We prove that a finitely generated equationally Noetherian group G is discriminating if and only if the quasivariety generated by G is the minimal universal class containing G. Among other results, we show that the non-abelian free nilpotent groups are non-discriminating. Finally we list some open problems concerning discriminating groups.

0 Introduction

This paper is concerned with the group-theoretic properties of separation and discrimination. These properties play a role in several areas of group theory, in particular, the theory of group varieties and the theory of algebraic geometry over groups (see [18] and [3]).

The main purpose of the paper is to answer certain questions which arose from [2]. We prove that a finitely generated equationally Noetherian group G is discriminating if and only if the quasivariety generated by G coincides with the universal closure of G (the minimal universal class containing G). Finding axioms of universal theories of finitely generated groups from nilpotent or metabelian varieties is an extremely difficult problem. A description of discriminating groups in these varieties would shed some light on this problem. Among other results we prove that non-abelian free solvable and non-abelian free nilpotent groups are non-discriminating. Moreover, we show that in all known (to us) cases non-discrimination is related to some kind of commutative transitivity of elements in the group, i.e., commutativity of centralizers of a particular type.

The paper is organized into the following four sections. In Section 1 we give all necessary definitions and basic results. Here we also exhibit examples of discrimi-

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nating groups. Surprisingly, several important types of groups from different areas of group theory turn out to be discriminating. These include the torsion-free abelian groups, Thompson's group F, the derived subgroup of a Gupta–Sidki group, and many of the Grigorchuk groups of intermediate growth of type G_ω. In Section 2, we establish links between discriminating groups and universal theories, Section 3 contains results on free nilpotent groups, and finally, in Section 4, we list several open questions concerning discriminating groups.

1 Preliminaries

We start by listing some definitions and results given in [2] and [3].

Definition 1.1. A group H is *separated* by a group G if for each non-trivial element $h \in H$ there is a homomorphism $\phi_h : H \to G$ such that $\phi_h(h) \neq 1$. If each ϕ_h is an epimorphism we also say that H is *residually* G. The group H is *discriminated* by G if to every finite set $X \subset H$ of non-trivial elements of H there is a homomorphism $\phi_X : H \to G$ such that $\phi_X(h) \neq 1$ for all $h \in X$. If each ϕ_X is an epimorphism, H is also called *fully residually* G.

Definition 1.2. A group G is called *discriminating* if every group separated by G is discriminated by G.

It should be pointed out that there is a difference between our notion of discriminating groups and the classical definition in H. Neumann [18] (see [18, Definitions 17.21 and 17.22]). According to [18] if G is a group and V is the least variety containing G, then G is called *discriminating* if to every finite set of words $w_1(x_1, \dots, x_n), \dots, w_k(x_1, \dots, x_n)$, in finitely many variables $x_1, \dots, x_n$, such that none of the equations $w_1(x_1, \dots, x_n) = 1, \dots, w_k(x_1, \dots, x_n) = 1$ is a law in V there is a tuple $(g_1, \dots, g_n) \in G^n$ for which simultaneously $w_1(g_1, \dots, g_n) \neq 1, \dots, w_k(g_1, \dots, g_n) \neq 1$. It is not hard to show that if a group G is discriminating in the sense of Definition 1.2 above then G is discriminating in the sense of [18]. However there are groups (e.g., non-abelian free groups) which are discriminating in the sense of [18] but are not discriminating in the sense of Definition 1.2. If we say that a group G is discriminating, we shall always mean in the sense of Definition 1.2.

Although it is difficult to determine which groups are discriminating they can be characterized in the following very simple manner:

Criterion ([2]). *A group G is discriminating if and only if its direct square $G \times G$ is discriminated by G.*

Proof. The necessity is obvious. Indeed, $G \times G$ is separated into G by the canonical projections.

For the sufficiency, suppose that G discriminates $G \times G$. It follows easily (by induction on n) that G discriminates G^n for all positive integers n. Now if G separates

H and $h_1, \dots, h_k$ are finitely many non-trivial elements of H, then there are homomorphisms $\phi_i : H \to G$ $(1 \leqslant i \leqslant k)$ such that $\phi_i(h_i) \neq 1$. Taking $\phi = \phi_1 \times \cdots \times \phi_k$, and using the assumption that G discriminates G^k, yields the desired conclusion.

Corollary 1.3. *Let G be a discriminating group and α be a cardinal. Then the Cartesian power G^α of G is also discriminating.*

Proof. If α is finite then, as mentioned above, $G^\alpha \times G^\alpha$ is discriminated by G, hence it is discriminated by G^α, and thus G^α is discriminating. If α is an infinite cardinal, then $G^\alpha \times G^\alpha$ is isomorphic to G^α, in particular, it is discriminated by G^α, and therefore G^α is discriminating.

Now we discuss several examples of discriminating and non-discriminating groups.

Proposition 1.4. *Torsion-free abelian groups are discriminating.*

Proof. We use additive notation here. Suppose that $(a_1, b_1), \dots, (a_n, b_n)$ are finitely many non-trivial elements in $A \times A$ where A is a torsion-free abelian group. We must find a homomorphism $A \times A \to A$ which does not annihilate any of the (a_i, b_i). We use induction on n.

When $n = 1$ the result is trivially true since A separates $A \times A$.

Now suppose inductively that the result is true for $n = k$. By inductive hypothesis, if $(a_1, b_1), \dots, (a_{k+1}, b_{k+1})$ are non-trivial elements of $A \times A$, then there is a homomorphism $f : A \times A \to A$ such that $f(a_i, b_i) \neq 0$ for $i = 1, \dots, k$. Moreover, since A separates $A \times A$, there is a homomorphism $g : A \times A \to A$ with $g(a_{k+1}, b_{k+1}) \neq 0$. Thus since A is torsion-free (and hence roots, when they exist, are unique), for a sufficiently large integer N the element $\phi = f + Ng$ will not annihilate any of the $(a_1, b_1), \dots, (a_{k+1}, b_{k+1})$. Hence, by induction, we are finished.

The more interesting case then is when an abelian group has torsion. If an abelian group with torsion is discriminating then its torsion subgroup must be infinite (see Proposition 1.11). Baumslag, Myasnikov and Remeslennikov [2] have some partial results on the characterization of torsion abelian discriminating groups.

An example of a finitely presented non-abelian discriminating group is given by R. Thompson's group F. The group F is a torsion-free infinite-dimensional FP_∞ group and it can be regarded as the group of orientation-preserving piecewise linear homeomorphisms from the unit interval $[0, 1]$ to itself that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are powers of 2 (see [6]).

Proposition 1.5. *Thompson's group F has the property that its direct square embeds in it, i.e., $F \times F \hookrightarrow F$. Hence it is a finitely presented non-abelian discriminating group.*

Proof. Consider the subgroup of F consisting of those homeomorphisms $g \in F$ satisfying the following three conditions:

(1) $g(\frac{1}{2}) = \frac{1}{2}$;

(2) $g([0, \frac{1}{2}]) \subseteq [0, \frac{1}{2}]$;

(3) $g([\frac{1}{2}, 1]) \subseteq [\frac{1}{2}, 1]$.

Note that any element of F fixes the end-points of $[0, 1]$. Given an ordered pair $(f_1, f_2) \in F \times F$, one constructs an element g in the subgroup as follows. Define

$$g(x) = \begin{cases} \frac{1}{2} f_1(2x) & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{2}(1 + f_2(2x - 1)) & \text{if } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Then the map $(f_1, f_2) \mapsto g$ is a group isomorphism from $F \times F$ onto the subgroup of F described above.

Other examples of finitely generated non-abelian discriminating groups are given by the commutator subgroups of the Gupta–Sidki groups. Recall that for each prime p there is a Gupta–Sidki group $H = H_p$ which is a subgroup of the automorphism group of a rooted tree (see [10]). For a given p, $H = H_p$ is then a 2-generator infinite p-group. It can be shown the commutator subgroup H' of H has the property that $H' \times H' \hookrightarrow H'$. Hence H' discriminates $H' \times H'$ and is therefore discriminating.

Proposition 1.6. *Let $H = H_p$ be a Gupta–Sidki group. Then its commutator subgroup is discriminating.*

The last class of groups we give as examples of discriminating groups are the Grigorchuk groups G_ω. Let p be a prime and let $\omega : \mathbb{N} \to \{0, 1, \dots, p\}$ be an infinite sequence of integers $0, 1, \dots, p$. For each such sequence ω, Grigorchuk [9] defined a finitely generated group G_ω which has intermediate growth. These groups have the following two properties.

(1) G_ω is residually a finite p-group for every sequence ω.

(2) If every number from the set $\{0, 1, \dots, p\}$ occurs in ω infinitely many times, then G_ω contains a copy of every finite p-group as a subgroup.

Proposition 1.7. *Let $\omega : \mathbb{N} \to \{0, 1, \dots, p\}$ be an infinite sequence in which every number from the set $\{0, 1, \dots, p\}$ occurs infinitely many times. Then G_ω is discriminating.*

(The following proof evolved from discussions with R. Grigorchuk.)

Proof. Let $\omega : \mathbb{N} \to \{0, 1, \dots, p\}$ be an infinite sequence in which every number from the set $\{0, 1, \dots, p\}$ occurs infinitely many times. To prove that G_ω is discriminating it suffices to show that G_ω discriminates $G_\omega \times G_\omega$. Since G_ω is residually a finite p-group, for every finite subset $S \subset G_\omega$ there exists a finite p-group K and a homomorphism $\phi : G_\omega \to K$ such that $\phi(g) \neq 1$ for each $g \in S$. It follows that for every finite subset $T \subset G_\omega \times G_\omega$ there exists a finite p-group L and a homomorphism $\phi : G_\omega \times G_\omega \to L$ such that $\phi(g) \neq 1$ for any $g \in T$. By property (2) above, there exists an embedding $\psi : L \to G_\omega$. Hence the homomorphism $\phi \circ \psi : G_\omega \times G_\omega \to G_\omega$ discriminates the set T into G_ω. This shows that G_ω is discriminating.

Other non-abelian finitely generated examples of groups G where $G \cong G \times G$ and hence discriminating groups are given in [11] and [19]. These are infinitely presented as are the Gupta–Sidki groups and the Grigorchuk groups G_ω.

Proposition 1.8. *Any non-abelian free group F is non-discriminating.*

Proof. Indeed, let a and b be two non-commuting elements in F. Then in the group $F \times F$ the non-trivial element $(a, 1)$ commutes with non-commuting elements $(1, a)$, $(1, b)$. If F discriminates $F \times F$ then there exists a homomorphism $\phi : F \times F \to F$ such that $\phi(a, 1) \neq 1$ and $[\phi(1, a), \phi(1, b)] \neq 1$. This implies that the centralizer of the non-trivial element $\phi(1, a)$ in F is non-abelian, a contradiction.

The argument in Proposition 1.8 works for any non-abelian group in which centralizers of non-trivial elements are abelian. Recall that groups with abelian centralizers of non-trivial elements are called *commutative transitive* (abbreviated CT). Discussions of groups of this type can be found in [7] and [20]. Observe that torsion-free hyperbolic groups are CT and subgroups of CT groups are CT. Now we have the following result.

Proposition 1.9. *A non-abelian CT group is non-discriminating.*

Example 1.10. Every non-trivial finite group is non-discriminating.

To see this, assume that K is a non-trivial finite discriminating group. Then K discriminates $K \times K$, hence there exists a monomorphism from $K \times K$ into K, which is impossible.

The argument in Example 1.10 provides the following more general result.

Proposition 1.11. *Let G be a group in which the non-trivial elements of finite order form a finite non-empty set. Then G is non-discriminating.*

The discussion above indicates, perhaps, that discriminating groups are close to abelian and far from hyperbolic. In what follows we discuss discriminating (or non-discriminating) groups in the varieties of abelian, nilpotent, and solvable groups.

Notice that Propositions 1.4 and 1.11 show that among finitely generated abelian groups only free abelian groups of finite rank are discriminating. As mentioned previously, Baumslag, Myasnikov and Remeslennikov have characterized only those torsion abelian groups which for each prime p the p-primary component modulo its maximal divisible subgroup contains no non-trivial elements of infinite p-height. The main question on which abelian groups are discriminating remains open (see the discussion at the end of this paper).

It is known that free solvable groups are CT ([16], [20]). This together with Proposition 1.9 gives the following result.

Proposition 1.12. *Non-abelian free solvable groups are non-discriminating, as well as their non-abelian subgroups.*

Theorem 17 of [20] asserts that if $G = A \operatorname{wr} B$, where A is an abelian group and B is a torsion-free abelian group, then G is CT. In the case where A is torsion-free this follows from the fact [5] that $A \operatorname{wr} B$ is universally equivalent to a non-abelian free metabelian group. (Here $A \neq 1$ and $B \neq 1$.) Hence we have the following

Corollary 1.13. *The restricted wreath product of two non-trivial torsion-free abelian groups is non-discriminating.*

Non-abelian nilpotent groups are not commutative transitive (since they have non-trivial center), so the argument above cannot be used directly to decide about non-discriminating nilpotent groups. Nevertheless, an extension of the commutative transitive property will be the main technique in showing that non-abelian free nilpotent groups are non-discriminating. This is one of the main results of the paper, proved in Section 3. We do not know whether there are any non-abelian finitely generated nilpotent discriminating groups.

2 Discriminating groups and logic

In this section we establish an important relation of discriminating groups with logic.

Let L be the first-order language with equality and a binary operation symbol $\cdot$, a unary operation symbol $^{-1}$, and a constant symbol 1. We call L the language of group theory. A *universal sentence* of L is one of the form $\forall x_1 \ldots \forall x_n\, \varphi(x_1, \ldots, x_n)$ where φ is a formula of L containing no quantifiers and containing at most the variables $x_1, \ldots, x_n$. It is easy to see that every universal sentence in the language L is logically equivalent to a formula of the following type:

$$\forall x_1 \ldots \forall x_n \left(\bigvee_j \left(\bigwedge_i (u_{ij}(x_1, \ldots, x_n) = 1 \right) \wedge \left(\bigwedge_k w_{kj}(x_1, \ldots, x_n) \neq 1) \right) \right),$$

where u_{ij}, w_{kj} are group words in the variables $x_1, \ldots, x_n$.

A class of groups $\mathcal{K}$ is *axiomatizable* by a set of universal sentences Σ in the language L if $\mathcal{K}$ consists precisely of all groups satisfying all formulas from Σ. In this event we say that $\mathcal{K}$ is *a universal class* and Σ is a set of *axioms* for $\mathcal{K}$. For a groups G denote by $\mathrm{Th}_\forall(G)$ the universal theory of G, i.e., the set of all universal sentences of L which are true in G. Two groups G and H are *universally equivalent* (and we write $G \equiv_\forall H$) if $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(H)$. The *universal closure* of G is the class $\mathrm{ucl}(G)$ axiomatizable by $\mathrm{Th}_\forall(G)$. Notice that $\mathrm{ucl}(G)$ is the minimal universal class containing G.

A *quasi-identity* in the language L is a formula of the type

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{i=1}^{m} r_i(x) = 1 \to s(x) = 1 \right), \tag{1}$$

where $r_i(x)$ and $s(x)$ are group words in $x_1, \dots, x_n$. A class of groups $\mathscr{K}$ is called a *quasivariety* if it can be axiomatized by a set of quasi-identities.

For a group G denote by $Q(G)$ the set of all quasi-identities which hold in G. Clearly $Q(G)$ is a set of axioms of the *minimal* quasivariety $\mathrm{qvar}(G)$ containing G.

It is convenient to have a purely algebraic characterization of the universal classes above. To this end, for a class of groups $\mathscr{K}$ we denote by $S(\mathscr{K})$, $P(\mathscr{K})$ and $P_u(\mathscr{K})$ the classes of all groups isomorphic to subgroups, unrestricted cartesian products and ultrapowers of groups from $\mathscr{K}$, respectively. It is known that $\mathrm{ucl}(G) = SP_u(G)$ (see, for example, [4] where this follows from Lemma 3.8 of Chapter 9). The quasivariety generated by G is the least axiomatic class containing G and closed under subgroups and unrestricted cartesian products. This class may be characterized as the class of all groups embeddable in a direct product of a family of ultrapowers of G. In symbols, $\mathrm{qvar}(G) = SPP_u(G)$; see [8]. We need one more class. If G is a group the least class containing G and closed under isomorphism, subgroups and direct products is the *prevariety* generated by G. This class may be realized as the class of all groups embeddable in a direct power of G. In symbols, $\mathrm{pvar}(G) = SP(G)$. In general, $\mathrm{pvar}(G)$ is not axiomatizable. Clearly

$$\mathrm{pvar}(H), \mathrm{ucl}(H) \subseteq \mathrm{qvar}(H).$$

Lemma 2.1. *If G is discriminating then every Cartesian power G^α is universally equivalent to G.*

Proof. Since G is embeddable into G^α we have $\mathrm{Th}_\forall(G^\alpha) \subseteq \mathrm{Th}_\forall(G)$ (universal sentences are preserved under taking subgroups). On the other hand, since G separates G^α, G discriminates G^α. This implies that $\mathrm{Th}_\forall(G^\alpha) \supseteq \mathrm{Th}_\forall(G)$. Indeed, if a universal sentence $\forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$ holds in G but does not hold in G^α then the negation $\neg\varphi$ of φ holds in G^α on some elements, say, $a_1, \dots, a_n$. Observe that $\neg\varphi(a_1, \dots, a_n)$ is equivalent to a finite system of equations and inequalities. Now there exists a homomorphism $\lambda : G^\alpha \to G$ which preserves all these inequalities (and equations). Therefore $\neg\varphi(\lambda(a_1), \dots, \lambda(a_n))$ holds in G, and this is a contradiction.

Under some circumstances the converse of the lemma above is also true. If G is a group and G_0 is a subgroup of G, then, given a word

$$w \in G_0 * \langle x_1, \dots, x_n; \rangle,$$

the equation $w = 1$ will be called an equation over G in $x_1, \dots, x_n$ with *coefficients* in G_0. In the case where $G_0 = 1$ the equation $w = 1$ will be called *coefficientless*. To formulate the statement that sometimes the converse of the lemma holds, we need to recall the following definition. A group G is called *equationally Noetherian* if every coefficientless system in finitely many variables is equivalent over G to a finite subsystem of itself. Notice that every abelian or linear group is equationally Noetherian. For a detailed discussion of equationally Noetherian groups, see [3]. Here we mention just the following

Theorem 2.2. ([3]) *Let G and H be finitely generated groups and let G be equationally Noetherian. Then G is universally equivalent to H if and only if G discriminates H and H discriminates G.*

This implies the following result.

Proposition 2.3. *Let G be a finitely generated equationally Noetherian group. Then G is discriminating if and only if G and $G \times G$ are universally equivalent.*

Since finitely generated nilpotent groups are linear (see [1]) we get the following corollary.

Corollary 2.4. *A finitely generated nilpotent group G is discriminating if and only if G is universally equivalent to $G \times G$.*

Lemma 2.5. *Let G and H be finitely generated groups. Let G be equationally Noetherian and discriminating. If H is universally equivalent to G then H is also discriminating.*

Proof. Suppose that $G \equiv_{\forall} H$. Then by Theorem 2.2, G discriminates H. Therefore $G \times G$ discriminates $H \times H$. Since G is discriminating, G discriminates $G \times G$. Again by Theorem 2.2, H discriminates G. This shows that H discriminates $H \times H$, and hence H is discriminating, as desired.

Now we formulate the main result of this section.

Theorem 2.6. *Let G be a finitely generated equationally Noetherian group. Then G is discriminating if and only if* $\mathrm{qvar}(G) = \mathrm{ucl}(G)$.

Proof. Suppose that G is discriminating. For a class of groups $\mathscr{K}$, we denote by $\mathscr{K}_\omega$ the subclass of all finitely generated groups from $\mathscr{K}$. To prove that $\mathrm{qvar}(G) = \mathrm{ucl}(G)$ it suffices to show that $\mathrm{qvar}(G)_\omega = \mathrm{ucl}(G)_\omega$. Indeed, this follows from the fact that every group is embeddable into an ultraproduct of its finitely generated subgroups. Since $\mathrm{qvar}(G) \supseteq \mathrm{ucl}(G)$ the inclusion $\mathrm{qvar}(G)_\omega \supseteq \mathrm{ucl}(G)_\omega$ is obvious. Notice now, that for an equationally Noetherian group G one has $\mathrm{qvar}(G)_\omega = \mathrm{pvar}(G)_\omega$ (see [17]), and therefore it suffices to show that $\mathrm{pvar}(G)_\omega \subseteq \mathrm{ucl}(G)_\omega$. Let H be a finitely generated group from $\mathrm{pvar}(G)$. Then $H \leqslant G^\alpha$ for some cardinal α. By Lemma 2.1 we have $G \equiv_{\forall} G^\alpha$, so that $G^\alpha \in \mathrm{ucl}(G)$. This implies that $H \in \mathrm{ucl}(G)$ because universal classes are closed under taking subgroups. This shows that $\mathrm{pvar}(G)_\omega \subseteq \mathrm{ucl}(G)_\omega$, as desired.

Now suppose that $\mathrm{qvar}(G) = \mathrm{ucl}(G)$. Then $G \times G \in \mathrm{ucl}(G)$, and thus $G \times G$ satisfies all the universal sentences which are true in G. On the other hand G is a subgroup of $G \times G$ and hence G satisfies all universal sentences which hold in $G \times G$. It follow that $G \equiv_\forall G \times G$. By Proposition 2.3, G is discriminating.

3 Discrimination of nilpotent groups

In this section we consider whether or not nilpotent groups are discriminating. To fix notation, we define left-normed commutators by induction as follows:

$$[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2;$$
$$[x_1, x_2, \dots, x_n] = [[x_1, x_2, \dots, x_{n-1}], x_n].$$

We denote by $Z_n(G)$ the nth term of the upper central series of G for $n = 0, 1, 2, \dots$. A group G is nilpotent of class c if c is the least integer for which $G = Z_c(G)$. Notice that for an element $g \in G$ the following equivalence holds: $g \notin Z_m(G)$ if and only if there exist $w_1, \dots, w_m \in G$ such that $[g, w_1, \dots, w_m] \neq 1$.

For subgroups A, B of a group G we write $[A, B]$ for the subgroup generated by all commutators $[a, b]$ with $a \in A$, $b \in B$. We denote by $\gamma_n(G)$ the nth term of the lower central series of G for $n = 1, 2, 3, \dots$. A group G is nilpotent of class c if and only if c is the least integer such that $\gamma_{c+1}(G) = 1$.

If G is any nilpotent group and x is any non-trivial element of G, then we let the *weight* of x, denoted by $\mathrm{wt}(x)$, be the unique integer n such that $x \in \gamma_n(G)$ but $x \notin \gamma_{n+1}(G)$. We let $\mathrm{wt}(1) = \infty > n$ for all integers n.

We let $F_r(\mathcal{N}_c)$ denote the free group of rank r in the variety of groups nilpotent of class at most c. This is the group $F_r/\gamma_{c+1}(F_r)$, where F_r is the absolutely free group of rank r.

As a first reduction, we can restrict to torsion-free nilpotent groups via the next theorem.

Theorem 3.1. *Any finitely generated nilpotent group with non-trivial torsion is non-discriminating.*

Proof. Suppose that G is a finitely generated nilpotent group with torsion subgroup T. Then T is finite since G satisfies the maximal condition for subgroups. The result then follows from Proposition 1.10.

Next we extend the idea of commutative transitivity.

Definition 3.2. A group G is *commutative transitive of level m* if G has the following property:

$$[x, y] = 1 \ \& \ [z, y] = 1 \ \& \ y \notin Z_m(G) \Rightarrow [x, z] = 1,$$

i.e., the centralizers of elements not in $Z_m(G)$ are abelian.

Observe that commutative transitive groups of level 0 are precisely the CT groups defined in Section 2; commutative transitive groups of level 1 are the commutation transitive groups which have been studied in [13].

The next result shows that commutative transitive groups of a given level m form a universal class.

Lemma 3.3. *A group G is commutative transitive of level m if and only if G satisfies the following universal sentence*:

$$\sigma_m = \forall x \forall y \forall z \forall w_1 \dots \forall w_m (((xy = yx) \wedge (yz = zy) \wedge ([y, w_1, \dots, w_m] \neq 1)) \\ \rightarrow (xz = zx)).$$

The proof follows directly from the definition.

Proposition 3.4. *Let G be a non-abelian commutative transitive group of level m. If G is not nilpotent of class $\leqslant m$ then G is non-discriminating.*

Proof. Assume to the contrary that G is discriminating. Then G is universally equivalent to $G \times G$ (by Lemma 2.1). This implies that $G \times G$ is also commutative transitive of level m. Since G is not nilpotent of nilpotency class $\leqslant m$, there exists $y \in G$ such that $y \notin Z_m(G)$, and hence $(y, 1) \notin Z_m(G \times G)$. Observe that the centralizer of y in $G \times G$ is non-abelian (indeed it contains $1 \times G$), and therefore $G \times G$ is not commutative transitive of level m.

Proposition 3.5. *A free nilpotent group of class c is commutative transitive of level $c - 1$.*

The proof follows from the following three lemmas:

Lemma 3.6 (cf. [12, Proposition 5.1]). *Let r and c be integers with* $\min\{r, c\} \geqslant 2$. *Let $G = F_r(\mathcal{N}_c)$ and $x, y \in G \backslash \{1\}$. Then* $[x, y] = 1$ *if and only if either* (1) $\mathrm{wt}(x) + \mathrm{wt}(y) \geqslant c + 1$; *or* (2) $\mathrm{wt}(x) = \mathrm{wt}(y) = n < \frac{1}{2}(c + 1)$ *and there exists an element $v \in G$ with* $\mathrm{wt}(v) = n$ *and there also exist integers $(p, q) \in (\mathbb{Z} \backslash \{0\})^2$ and elements $(z_1, z_2) \in G^2$ such that simultaneously $x = v^p z_1$, $y = v^q z_2$, and*

$$\min\{\mathrm{wt}(z_1) + n, \mathrm{wt}(z_2) + n, \mathrm{wt}(z_1) + \mathrm{wt}(z_2)\} \geqslant c + 1.$$

This result follows directly from results of Magnus (see [14] or [15]). The result seems to be known but has never appeared in print in this form. An equivalent version describing centralizers in free nilpotent groups was given in [12].

The following are straightforward consequences of the characterization of commutativity in terms of weights.

Lemma 3.7. *Let r and c be integers with* $\min\{r, c\} \geqslant 2$. *Let $G = F_r(\mathcal{N}_c)$. Then an element $y \in G$ has* $\mathrm{wt}(y) = 1$ *if and only if there exist $w_1, \dots, w_{c-1} \in G$ such that $[y, w_1, \dots, w_{c-1}] \neq 1$ in G.*

Lemma 3.8. *Let $G = F_r(\mathcal{N}_c)$ and suppose that $r, c > 1$. Let $x, y, z \in G \backslash \{1\}$ be such that* $\mathrm{wt}(y) = 1$, $[x, y] = 1$ *and* $[y, z] = 1$. *Then* $[x, z] = 1$.

The proof of Theorem 3.1 is now direct. Suppose that $x, y \in F_r(\mathcal{N}_c)$ for some $r, c > 1$ and suppose that $[x, y] = 1$, $[y, z] = 1$ and there exist $w_1, \ldots, w_{c-1} \in F_r(\mathcal{N}_c)$ with $[y, w_1, \ldots, w_{c-1}] \neq 1$. Thus from Lemma 3.7 we have $\mathrm{wt}(y) = 1$. If $\mathrm{wt}(x) \neq 1$ then from Lemma 3.6 we have $\mathrm{wt}(x) \geqslant c$. Then $\mathrm{wt}(x) + \mathrm{wt}(z) \geqslant c + 1$ and $[x, z] = 1$, again from Lemma 3.6. The analogous fact is true if $\mathrm{wt}(z) \neq 1$.

Therefore we can reduce to the case where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$. The result then follows from Lemma 3.8.

Theorem 3.1. *Every non-abelian free nilpotent group is non-discriminating.*

The theorem follows immediately from Propositions 3.4 and 3.5 since a non-abelian free nil-c group has class c.

4 Open Questions

In this final section, we list several open problems on discriminating groups.

Question D1. Describe in terms of Ulm and Szmielew invariants the abelian discriminating groups.

Question D2. Are there any non-abelian finitely generated nilpotent discriminating groups? In particular, are the unitriangular nilpotent groups $\mathrm{UT}_n(\mathbb{Z})$ discriminating? We note that $\mathrm{UT}_4(\mathbb{Z})$ is non-discriminating.

Question D3. What (if any) are the finitely generated metabelian discriminating groups?

Remark 4.1. If $G \times G \hookrightarrow G$, then G is discriminating. In particular, if $G \times G \cong G$, then G is discriminating.

A great deal of work has been done on the question of when $G \times G \cong G$. This is more interesting under the hypothesis of finite generation since, for example, if G_0 is any non-trivial group and I is any infinite index set then $G = G_0^I$ satisfies $G \cong G \times G$. There are finitely generated examples of groups $G \neq 1$ (necessarily non-solvable) that satisfy $G \cong G \times G \cong G$ (see [11] and [19]) but no known finitely presented examples (see [11]).

Question D4 (Peter Hilton). Do there exist non-trivial finitely presented groups G that satisfy $G \cong G \times G$?

At present we know only two types of examples of discriminating groups: abelian groups and groups embeddable into their Cartesian square.

Question D5. Suppose that G is a finitely presented discriminating group. If $G \times G$ does not embed in G must G be abelian?

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