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THREE-DIMENSIONAL 2-FRAMED TQFTS AND SURGERY

STEPHEN F. SAWIN

Abstract. The notion of 2-framed three-manifolds is defined. The category of 2-framed cobordisms is described, and used to define a 2-framed three-dimensional TQFT. Using skeletonization and special features of this category, a small set of data and relations is given that suffice to construct a 2-framed three-dimensional TQFT. These data and relations are expressed in the language of surgery.

Introduction

The work of Jones [Jon85], Witten [Wit89], Atiyah [Ati89] and of Reshetikhin and Turaev [RT90, RT91] set the stage for how we have approached the Chern-Simons (or Jones-Witten or Reshetikhin-Turaev) invariants and their cousins ever since. The 'natural definition' of the invariants is as the partition function of a quantum gauge theory with the Chern-Simons functional as action, computed formally by a mathematically inaccessible path integral. Witten argued that these invariants could be effectively computed by techniques of physics, chief among them the cut-and-paste properties expected of the partition function of a topological quantum field theory. Atiyah translated these cut-and-paste properties into a precise set of axioms which one could prove a rigorously defined invariant satisfies, thus gaining mathematically respectable access to some of the path integral arguments that are so famously effective in the hands of physicists. The term 'topological quantum field theory' or TQFT is generally used in the mathematical literature to refer to anything which satisfies these axioms, though of course they capture only a portion of the tools physicists use to understand TQFTs. Finally Reshetikhin and Turaev used quantum group techniques to construct a rigorous topological invariant corresponding to Chern-Simons theory with gauge group $SU(2)$ and showed that it satisfied a weaker version of Atiyah's axioms (in the language of this paper, they showed everything but the mending axiom).

The current state of the art is that from every modular category one can construct a TQFT (See Turaev [Tur94]). A variety of interesting invariants fit into this framework, most of them closely related to Chern-Simons theory and, like Chern-Simons theory, are three-dimensional and involve an additional topological structure called a 2-*framing* or *biframing.* Many invariants do not fit this framework, but appear to be analogous in some sense, such as those of Hennings [Hen96], Lyubashenko [Lyu95b, Lyu95a] and Kuperberg [Kup96] (see also related work of Kerler [Ker95, Ker97, Ker98] and Kauffman and Radford [KR95]).

Because the notion of TQFT is fairly involved, and the topology and algebra required to discuss it is complex (though certainly not deep), it is often easier to deal with these invariants purely as three-manifold invariants, ignoring the underlying TQFT structure. Likewise, it is easier to ignore the technicalities of the 2-framing by including a correction term to make the invariant of closed manifolds independent of the 2-framing and treating the associated representation of the 2-framed mapping class group as a projective representation of the ordinary mapping class group. Such simplifications, however, leave out important topological information, and perhaps more importantly, sidestep some of the best connections available between the well worked out topology and combinatorics of constructing invariants and the deep but very poorly understood geometry and physics underlying these invariants.

This article reduces the definition of a three-dimensional 2-framed TQFT to a small set of relatively straightforward conditions to check, and then expresses these conditions in the language of surgery (many of the invariants are computed and defined most naturally in terms of surgery presentations of 2-framed three-manifolds). There are two motivations for this. The first is that these conditions offer simplifications and fresh insight in the construction of TQFTs associated to the Reshetikhin-Turaev invariants, and more generally constructing TQFTs from modular categories. In particular, [Saw96] gives a relatively brief self-contained construction of TQFTs from modular categories using these techniques. In fact the main results here are used in that paper with a reference to unpublished lecture notes, and the present article offers a published reference for them. The second motivation is as a framework for generalizations of the notion of TQFT, such as adding spin and other structures, extending the theory to encompass larger codimensions as in Walker [Wal], and weakening the TQFT structure to apply to the Hennings and Kuperberg invariants.

The first section reviews the notion of a 2-framing of a three-manifold, generalizes it to manifolds with boundary, constructs the symmetric

monoidal category of 2-framed cobordisms, and defines a TQFT in terms of this category. The second section simplifies this definition by general category theoretic arguments and use of structure special to the 2-framed cobordism category. The third section translates this simplified set of conditions into the language of surgery, in terms of which many of the interesting invariants are defined.

1. 2-Framings and the 2-Framed Cobordism Category

1.1. **2-Framings.** Atiyah [Ati90] defined the notion of a 2-framing and analyzed its properties, identifying it as the anomaly in Witten's construction of Chern-Simons theory. Several different but equivalent formulations of this anomaly have been given, most centering either on bundles over a three manifold as in Atiyah's original definition or on a choice of four-manifold which the three-manifold bounds. We follow the second approach, because of the simplicity of the definition, the underlying four-dimensional nature of Chern-Simons theory (see Dijkgraaf and Witten [DW90]), and the surgery description. Our approach follows most closely Walker [Wal]. The chief disadvantage is the awkwardness of viewing a four-dimensional category as a three-dimensional category with extra structure.

All of our manifolds will be assumed to be compact, oriented, and smooth, but not necessarily connected or closed (although we will usually add the term "with boundary" when speaking of manifolds which may not be closed).

A 2-*framed three-manifold* is a four-manifold M with boundary, with the proviso that each component of M have at most one component of boundary (in fact since we will only really be concerned with our four-manifolds up to cobordism, this does not affect the theory, but makes a number of technicalities easier to deal with). We will refer to M as a choice of 2-framing on its boundary ∂M. A 2-*framed diffeomorphism* between 2-framed three-manifolds M and N is a diffeomorphism $f\ \partial M \to \partial N$ together with a five-manifold W whose boundary is $M \cup_f N$, the closed manifold formed by identifying ∂M with ∂N via f. Thus [Kir89] two 2-framed three-manifolds are 2-framed diffeomorphic if and only if their boundaries are diffeomorphic and they have the same signature.

A 2-*framed surface* Σ is a three-manifold with boundary (again with the proviso that each component has connected boundary) , and we will refer to Σ as a choice of 2-framing on $\partial\Sigma$. A 2-*framed diffeomorphism* of 2-framed surfaces is just a diffeomorphism of the 2framed surfaces as three-manifolds. A 2-*framed three-manifold with boundary* (M, Σ) is a 2-framed three-manifold M together with a 2-framed surface Σ which

as a three-manifold is a submanifold of the boundary ∂M. We will refer to (M, Σ) as a choice of 2-framing on $\partial M - \Sigma$. A *diffeomorphism of 2-framed manifolds* (M, Σ) and (N, Γ) with boundary is a diffeomorphism of the 2-framed three manifolds whose underlying ordinary diffeomorphism of the boundaries sends Σ to Γ.

If (M, Σ) and (N, Γ) are 2-framed manifolds with boundary and f is a 2-framed diffeomorphism of subsets of their boundaries $\Sigma' \subset \Sigma$ and $\Gamma' \subset \Gamma'$, then we can we can form the gluing of these, $M \cup_f N$, which is the 2-framed three-manifold with boundary which as a four manifold is M and N glued together along the subsets Σ' and Γ' of their boundary, and whose boundary 2-framed surface is the subset $\Sigma - \Sigma' \cup \Gamma - \Gamma'$. See Figure 1 for a pictorial representation of gluing (here for visual clarity all dimensions have been reduced by one, so that M and N which should be represented as four-manifolds are pictured as three-manifolds, Σ and Γ which should be represented by three-manifolds are pictured as two-manifolds, etc.

Notice if (M, Σ) and (N, Γ) are 2-framed three-manifolds with boundary, then Σ is a choice of 2-framing on the boundary of ∂M, Γ is a choice of 2-framing on the boundary of ∂N, and $M \cup_f N$ is a choice of 2-framing on $(\partial M - \Sigma) \cup_{\partial f} (\partial N - \Gamma)$. Thus roughly speaking the notion of "a 2-framing on" commutes with the operations of taking the boundary and gluing.

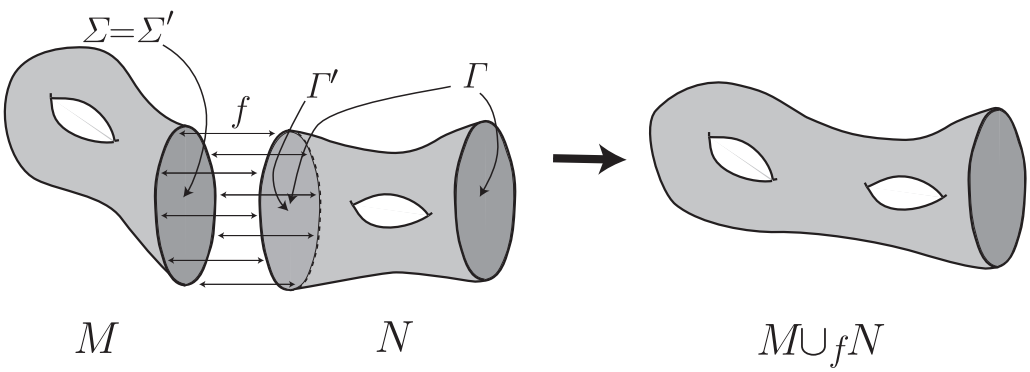

FIGURE 1. The gluing of M and N

Below we shall refer to various notions for 2-framed manifolds and diffeomorphisms, such as disjoint union and orientation reversal, which are defined exactly as for ordinary manifolds and diffeomorphisms, and we will not bother to state the definition precisely.

1.2. **The 2-framed cobordism category.** Let us define a 2-framed cobordism as $\mathfrak{m} = (M, \Sigma, \Gamma, f)$, where M is a 2-framed three-manifold

with boundary, Σ and Γ are 2-framed two-manifolds, and f is an orientation preserving 2-framed diffeomorphism of $\Sigma^* \cup \Gamma$ to the boundary of M (here Σ^* means Σ equipped with the opposite orientation). We say that two 2-framed cobordisms $\mathfrak{m} = (M, \Sigma, \Gamma, f)$ and $\mathfrak{m}' = (M', \Sigma, \Gamma, f')$ are the same if there is a 2-framed diffeomorphism $F\colon M \to M'$ such that $F \circ f = f'$ (in particular, they are the same if and only if they are 2-framings of diffeomorphic three-dimensional ordinary cobordisms and their underlying four-manifolds have the same signature). If $\mathfrak{m} = (M, \Sigma, \Gamma, f)$ we shall simply write $\mathfrak{m}\colon \Sigma \to \Gamma$ in the frequent situation that we do not need to refer explicitly to the underlying manifold and parameterization map f.

If $\mathfrak{m} = (M, \Sigma, \Gamma, f)$ and $\mathfrak{n} = (N, \Gamma, \Delta, g)$ are two 2-framed cobordisms, then $h = g|_\Gamma \circ (f|_\Gamma)^{-1}$ is an orientation-reversing 2-framed diffeomorphism between subsets of the boundary of M and N, and thus we get a 2-framed three-manifold $M \cup_h N$ and a 2-framed cobordism, the *composition* of $\mathfrak{m}$ and $\mathfrak{n}$ defined as

$$\mathfrak{n} \circ \mathfrak{m} = (M \cup_h N, \Sigma, \Delta, f|_{\Sigma^*} \cup g|_\Delta).$$

If Σ is a 2-framed two-manifold, $(\Sigma \times I, \Sigma^* \times \{0\} \cup \Sigma \times \{1\})$ is of course a 2-framed three-manifold with boundary. This determines a 2-framed cobordism

$$\mathbf{1}_\Sigma = (\Sigma \times I, \Sigma, \Sigma, f)$$

where f sends the first Σ to $\Sigma \times \{0\}$ and the second to $\Sigma \times \{1\}$, both by the identity map. One readily sees that

$$(\mathfrak{m} \circ \mathfrak{n}) \circ \mathfrak{p} = \mathfrak{m} \circ (\mathfrak{n} \circ \mathfrak{p})$$

$$\mathbf{1}_\Sigma \circ \mathfrak{m} = \mathfrak{m} \circ \mathbf{1}_\Sigma = \mathfrak{m}$$

whenever the compositions are well-defined.

If we take the somewhat pedantic position that the empty three-manifold $\emptyset$ is a 2-framed two-manifold and the empty map $\emptyset$ is a 2-framed diffeomorphism from $\emptyset$ to $\emptyset$, we can view closed 2-framed three-manifolds M as cobordisms via $(M, \emptyset, \emptyset, \emptyset)$. For a slight additional cost in pedantry, we may also view $\emptyset$ as a 2-framed three-manifold, allowing us to shamelessly write $\mathbf{1}_\emptyset = (\emptyset, \emptyset, \emptyset, \emptyset)$. The relations above still hold for these vacuous cobordisms.

If $\mathfrak{m} = (M, \Sigma_1, \Sigma_2, f)$ and $\mathfrak{n} = (N, \Gamma_1, \Gamma_2, g)$ are two 2-framed cobordisms, the disjoint union is

$$\mathfrak{m} \cup \mathfrak{n} = (M \cup N, \Sigma_1 \cup \Gamma_1, \Sigma_2 \cup \Gamma_2, f \cup g).$$

Again it is clear that

$$\mathbf{1}_\emptyset \cup \mathfrak{m} = \mathfrak{m} \cup \mathbf{1}_\emptyset = \mathfrak{m},$$

$$(\mathfrak{m} \cup \mathfrak{n}) \cup \mathfrak{p} = \mathfrak{m} \cup (\mathfrak{n} \cup \mathfrak{p}) \text{ and}$$

$$(\mathfrak{m} \cup \mathfrak{n}) \circ (\mathfrak{m}' \cup \mathfrak{n}') = (\mathfrak{m} \circ \mathfrak{m}') \cup (\mathfrak{n} \circ \mathfrak{n}').$$

If Σ and Γ are any two 2-framed two-manifolds define

$$\sigma_{\Sigma,\Gamma} = ((\Sigma \cup \Gamma) \times I, \Sigma \cup \Gamma, \Gamma \cup \Sigma, f)$$

where f maps $\Sigma \cup \Gamma$ to $(\Sigma \cup \Gamma) \times \{0\}$ by the identity map and and $\Gamma \cup \Sigma$ to $(\Sigma \cup \Gamma) \times \{1\}$ by the obvious flip of the identity. Of course

$$\sigma_{\Sigma,\Gamma} \circ (\mathfrak{m} \cup \mathfrak{n}) \circ \sigma_{\Gamma,\Sigma} = \mathfrak{n} \cup \mathfrak{m} \text{ and}$$

$$(\mathbf{1}_\Gamma \cup \sigma_{\Sigma,\Delta}) \circ (\sigma_{\Sigma,\Gamma} \cup \mathbf{1}_\Delta) = \sigma_{\Sigma,\Gamma\cup\Delta}.$$

This is all to say that there is a category whose objects are 2-framed two-manifolds and whose morphisms are 2-framed cobordisms with composition defined above and $\mathbf{1}_\Sigma$ as identity, that $\cup$ forms a strict monoidal product on this category with $\emptyset$ as trivial object, and that σ forms a symmetry for this monoidal category. We call this symmetric monoidal category the 2-*framed cobordism category* $\mathfrak{BC}$.

$\mathfrak{BC}$ has some particularly nice properties, revolving around the artificiality of dividing up the boundary into two pieces. In particular to each 2-framed two-manifold Σ we can associate Σ^*, which is Σ with the opposite orientation, and cobordisms

$$\mathfrak{d}_\Sigma = (\Sigma \times I, \Sigma \cup \Sigma^*, \emptyset, f) \text{ and}$$

$$\mathfrak{e}_\Sigma = (\Sigma \times I, \emptyset, \Sigma^* \cup \Sigma, f)$$

where in both cases f is the obvious map from $\Sigma^* \cup \Sigma$ to the boundary of $\Sigma \times I$. These morphisms have the properties

$$(\mathfrak{d}_\Sigma \cup \mathbf{1}_\Sigma) \circ (\mathbf{1}_\Sigma \cup \mathfrak{e}_\Sigma) = \mathbf{1}_\Sigma \text{ and}$$

$$(\mathbf{1}_{\Sigma^*} \cup \mathfrak{d}_\Sigma) \circ (\mathfrak{e}_\Sigma \cup \mathbf{1}_{\Sigma^*}) = \mathbf{1}_{\Sigma^*}.$$

A monoidal category which admits such morphisms is called a *rigid monoidal category.* A rigid symmetric monoidal category is sometimes referred to in the literature as a closed category or a tensor category, though the latter term is often used as a synonym for monoidal category.

Remark 1. The reader may note the asymmetry of the treatment of the monoidal and symmetric structure on the one hand, which are part of the definition of $\mathfrak{BC}$, and the rigidity on the other, which is observed after the fact to be a structure that can be assigned to the category. When we construct functors from this category to other categories, the target categories will have explicit symmetric monoidal structures, and we will want the functors to make these structures correspond exactly, but will not ask this of the rigidity. The reason is that while the symmetry and rigidity are both quite canonical in this category, in the

linear category to which we will map, the rigidity is distinctly fussier than the symmetry.

1.3. **2-Framed TQFTs.** The category Vect whose objects are finite-dimensional vector spaces and whose morphisms are linear maps with the usual composition can also be made into a symmetric monoidal category. The monoidal structure sends vector spaces V and W to the tensor product $V \otimes W$ and linear maps $f\colon V \to V'$ and $g\colon W \to W'$ to the tensor product $f \otimes g\colon V \otimes W \to V' \otimes W'$, and makes the ground field the trivial object. The symmetric morphism $\sigma_{v,w}$ sends the vector $v \otimes w$ to $w \otimes v$.

A symmetric monoidal functor between two symmetric monoidal categories is an ordinary functor which preserves the symmetry and monoidal structure.

Remark 2. A more careful treatment is to assume both categories are *weak* monoidal categories. For instance, instead of asserting that vector spaces $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ are equal, we should provide a well-behaved isomorphism between them. Any careful attempt to define either union of manifolds or tensor product of vector spaces requires this. Nevertheless, it is the common and generally harmless custom in mathematics to view the vector spaces $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ as "the same," and in the interest of clarity and simplicity we will continue this relaxed practice, pointing out occasional subtleties.

Definition 1. A TQFT is a symmetric monoidal functor from $\mathfrak{BC}$ to Vect .

Specifically, a TQFT Z assigns a vector space $Z(\Sigma)$ to each 2-framed two-manifold Σ, and a linear map $Z(\mathfrak{m})\colon Z(\Sigma) \to Z(\Gamma)$ to each 2-framed cobordism $\mathfrak{m}\colon \Sigma \to \Gamma$, satisfying the following conditions:

1. $Z(\mathfrak{m}) \circ Z(\mathfrak{n}) = Z(\mathfrak{m} \circ \mathfrak{n})$
2. $Z(\mathbf{1}_\Sigma) = \mathbf{1}_{Z(\Sigma)}$
3. $Z(\Sigma \cup \Gamma) = Z(\Sigma) \otimes Z(\Gamma)$
4. $Z(\mathfrak{m} \cup \mathfrak{n}) = Z(\mathfrak{m}) \otimes Z(\mathfrak{n})$
5. $Z(\emptyset) = \mathbb{F}$, the ground field
6. $Z(\sigma_{\Sigma,\Gamma}) = \sigma_{Z(\Sigma),Z(\Gamma)}$.

Two TQFTs are *equivalent* if there is a natural isomorphism between them. That is, for each 2-framed surface Σ, an isomorphism $i_\Sigma\colon Z_1(\Sigma) \to Z_2(\Sigma)$ such that if $\mathfrak{m}\colon \Sigma \to \Gamma$ then $Z_2(\mathfrak{m}) \circ i_\Sigma = i_\Gamma \circ Z_1(\mathfrak{m})$ and $i_{\Sigma\cup\Gamma} = i_\Sigma \otimes i_\Gamma$.

Remark 3. Notice that a TQFT offers in particular an invariant of cobordisms from $\emptyset$ to $\emptyset$, which is to say an invariant of closed 2-framed

three-manifolds up to 2-framed diffeomorphism. This means an invariant of four-manifolds with boundary up to cobordism.

2. Characterization of TQFTs

2.1. **Skeletonization.** One problem with defining a TQFT is that we seem to have a huge array of objects, each of which would need to be assigned a vector space simply as a starting point. Of course it is clear that many of these are not fundamentally different, and there is much less significant choice involved then it would appear. This is a common situation in category theory, where the notion of isomorphism captures it precisely. Specifically, two objects Σ and Γ are *isomorphic* if there exist morphisms $\mathfrak{m}\colon \Sigma \to \Gamma$ and $\mathfrak{n}\colon \Gamma \to \Sigma$ such that $\mathfrak{m} \circ \mathfrak{n} = \mathbf{1}_\Gamma$ and $\mathfrak{n} \circ \mathfrak{m} = \mathbf{1}_\Sigma$.

Lemma 1. *Two 2-framed surfaces are isomorphic as objects in $\mathfrak{BC}$ if and only if they are 2-framings of diffeomorphic 2-manifolds.*

Proof. Of course, an isomorphism between two 2-framed surfaces must be a 2-framing of an isomorphism between two ordinary surfaces in the category of ordinary three-dimensional cobordisms, and thus the surfaces must be diffeomorphic.

Suppose Σ and Γ are 2-framed surfaces which are 2-framings of the same surface S (here we have absorbed the ordinary diffeomorphism of the surfaces for notational simplicity). Viewing Σ and Γ as three-manifolds with boundary S, we form the closed three-manifold $\Sigma^* \cup_S (S \times I) \cup_S \Gamma$ and choose some four-manifold W which it bounds. Then $(W, \Sigma^* \cup \Gamma)$ is a 2-framed three-manifold with boundary and $\mathfrak{n} = (W, \Sigma, \Gamma, f)$ is a cobordism from Σ to Γ with the obvious f. Likewise glue $\Gamma^* \cup_S (S \times I) \cup_S \Sigma$ and choose a 2-framing to get a 2-framed cobordism $\mathfrak{n}'\colon \Gamma \to \Sigma$. Now $\mathfrak{n}' \circ \mathfrak{n}$ represents a 2-framing on $S \times I$, and thus is the same 2-framed cobordism as $\mathbf{1}_\Sigma$ if and only if it has the same signature (here by the signature we mean the signature of the four-manifold which forms the 2-framed manifold underlying the 2-framed cobordism). Now by Wall's nonadditivity of the signature result [Wal69] the signature of $\mathfrak{n}' \circ \mathfrak{n}$ is not the sum of the signatures of $\mathfrak{n}$ and $\mathfrak{n}'$, but the difference depends only on the boundaries, so by connect summing an appropriate number of copies of $\pm CP^2$ to $\mathfrak{n}'$ we get a new cobordism $\mathfrak{n}^{-1}$ such that $\mathfrak{n}^{-1} \circ \mathfrak{n}$ has the same signature as, and thus is the same morphism as, $\mathbf{1}_\Sigma$. Therefore Σ and Γ are isomorphic. $\square$

Thus there is one isomorphism class of objects for each genus.

Recall that two (symmetric, monoidal) categories $\mathfrak{C}$ and $\mathfrak{C}'$ are *equivalent* if there are a pair of (symmetric, monoidal) functors F, G between them such that $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity.

Proposition 1. *For any choice of a 2-framed surface for each genus, $\mathfrak{BC}$ is equivalent to the full subcategory $\mathfrak{BC}'$ formed from arbitrary unions of these surfaces (i.e., arbitrary unions of these surfaces are the objects, all morphisms between them form the morphisms).*

Proof. The functor from $\mathfrak{BC}'$ to $\mathfrak{BC}$ is just the inclusion functor. For the other direction choose for each object of $\mathfrak{BC}$ which is connected an isomorphism to the chosen object of the same genus (using Lemma 1). An arbitrary object Σ of $\mathfrak{BC}$ is a union of connected pieces, so it is the domain of an isomorphism $\mathfrak{i}_\Sigma$ which is a union of the chosen isomorphisms. Then the functor sends Σ to the isomorphic object in $\mathfrak{BC}'$ and a morphism $\mathfrak{m}\colon \Sigma \to \Gamma$ to $\mathfrak{i}_\Gamma \circ \mathfrak{m} \circ \mathfrak{i}_\Sigma^{-1}$. The composition of functors one way is the identity, and the other way is isomorphic to the identity via the natural isomorphism $\mathfrak{i}_\Sigma^{-1}$. □

Remark 4. Typically the so-called skeletonization process of the previous proposition results in a weak monoidal structure. In this case the monoidal structure is free, in the sense that two-manifolds and cobordisms can be uniquely written as the union of connected components (up to symmetry) and because of this there is no weakening of the structure.

Corollary 1. *Every symmetric monoidal functor from the full subcategory $\mathfrak{BC}'$ given above to* Vect *uniquely determines a TQFT up to equivalence.*

Proof. Let F be the equivalence functor $\mathfrak{BC} \to \mathfrak{BC}'$, and Z_0 be the functor $\mathfrak{BC}' \to \mathrm{Vect}$. Then of course $Z_0 \circ F$ is a symmetric monoidal functor $\mathfrak{BC} \to \mathrm{Vect}$, and hence a TQFT. If Z is a TQFT whose restriction to $\mathfrak{BC}'$ is Z_0, and $\mathfrak{i}_\Sigma$ is the natural isomorphism from F to the identity functor on $\mathfrak{BC}$, then $Z(\mathfrak{i}_\Sigma)$ is an isomorphism from $Z_0 \circ F(X)$ to $Z(X)$. It is natural because $\mathfrak{i}_\Sigma$ is natural, and thus Z and $Z_0 \circ F$ are equivalent. □

2.2. **Minimal data and relations.** There is still quite a bit of redundancy involved in checking that something is a TQFT. In particular, the division of the parameterized boundary into source and target seems quite arbitrary: All such divisions are the same in some sense, and it seems that once you know the value of the TQFT on one of these you know them all. This subsection makes that notion precise.

Consider the set of all *basic* cobordisms: i.e. 2-framed cobordisms $\mathfrak{m}\colon \bigcup_{i=1}^n \Sigma_{g_i} \to \emptyset$ with target the trivial surface, source a union of the chosen 2-framed surfaces, and whose underlying four-manifold M is connected.

Choose for each chosen Σ_g an orientation-reversing map $S\colon \Sigma_g \to \Sigma_g$ such that $S^2 = \mathrm{id}$ (the choice of identification of Σ_g with its orientation reversal is necessary in describing the rigidity structure because we have skeletonized: Making the identification an involution ($S^2 = \mathrm{id}$) is not, but allows us to keep track of fewer issues of ordering). Choose for each Σ_g duality morphisms $\mathfrak{d}_g = (\Sigma_g \times I, \Sigma_g \cup \Sigma_g^*, \emptyset, f)$ and $\mathfrak{e}_g = (\Sigma_g \times I, \emptyset, \Sigma_g^* \cup \Sigma_g, f)$ and compose the parameterization maps on the Σ_g^* pieces with S to get a *cap* morphism $\hat{\mathfrak{d}}_g$ and a *cup* morphism $\hat{\mathfrak{e}}_g$

$$\hat{\mathfrak{d}}_g\colon \Sigma_g \cup \Sigma_g \to \emptyset$$
$$\hat{\mathfrak{e}}_g\colon \emptyset \to \Sigma_g \cup \Sigma_g$$

such that

$$((\hat{\mathfrak{d}}_g \cup \mathbf{1}_g) \circ (\mathbf{1}_g \cup \hat{\mathfrak{e}}_g) = (\mathbf{1}_g \cup \hat{\mathfrak{d}}_g) \circ (\hat{\mathfrak{e}}_g \cup \mathbf{1}_g) = \mathbf{1}_g$$

where $\mathbf{1}_g$ is shorthand for $\mathbf{1}_{\Sigma_g}$. Notice that $\hat{\mathfrak{d}}_g$ is a basic cobordism.

We wish to consider three operations on basic cobordisms

Permuting: Suppose σ is a permutation of $(1, \dots, n)$. If $f\colon \bigcup_{i=1}^n \Sigma_{g_i} \to \partial M$ then σ acts on the left in an obvious way on f, $\sigma(f)(x_1, \dots, x_n) = f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$, and thus if $\mathfrak{m} = (M, \bigcup_{i=1}^n \Sigma_{g_i}, \emptyset, f)$ is basic we get a new basic cobordism $\sigma(\mathfrak{m}) = (M, \bigcup_{i=1}^n \Sigma_{g_{\sigma^{-1}(i)}}, \emptyset, \sigma(f))$ which of course is $\mathfrak{m}$ composed with an appropriate product of symmetry morphisms.

Sewing: Suppose $\mathfrak{m}\colon \bigcup_{i=1}^m \Sigma_{g_i} \to \emptyset$ and $\mathfrak{n}\colon \bigcup_{i=1}^n \Sigma_{g'_i} \to \emptyset$ with $g_m = g'_i$. Then we can form the *sewing* of $\mathfrak{m}$ and $\mathfrak{n}$

$$\mathfrak{m} \cup_s \mathfrak{n} = (\mathfrak{m} \cup \mathfrak{n}) \circ (\mathbf{1}_{g_1} \cup \dots \cup \mathbf{1}_{g_{m-1}} \cup \hat{\mathfrak{e}}_{g_m} \cup \mathbf{1}_{g'_2} \cup \dots \cup \mathbf{1}_{g'_n}).$$

This corresponds to gluing the underlying manifolds together via S along the Σ_{g_m} component of the boundaries. See Figure 2 for an illustration.

Mending Suppose $\mathfrak{m}\colon \bigcup_{i=1}^m \Sigma_{g_i} \to \emptyset$ and $g_1 = g_2$. Then we can form the *mending* of $\mathfrak{m}$

$$\mathfrak{m}_m = \mathfrak{m} \circ (\hat{\mathfrak{e}}_{g_1} \cup \mathbf{1}_{g_3} \cup \dots \cup \mathbf{1}_{g_m}).$$

This glues the underlying manifold to itself via S, thus increasing the dimension of the first homology. See Figure 2 for an illustration.

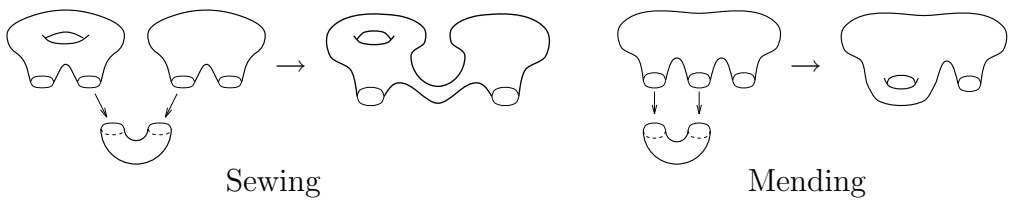

FIGURE 2. Mending versus sewing for ordinary two-manifolds with boundary

Theorem 1. *Suppose we are given for each chosen surface Σ_g a finite-dimensional vector space $Z(\Sigma_g)$, and for each basic morphism*

$$\mathfrak{m}\colon \bigcup_{i=1}^{n} \Sigma_{g_i} \to \emptyset$$

a functional $Z(\mathfrak{m})$ on $\bigotimes_{i=1}^{n} Z(\Sigma_{g_i})$ satisfying the following conditions (If $\mathfrak{m}\colon \emptyset \to \emptyset$ we understand $Z(\mathfrak{m})$ is a functional on the ground field $\mathbb{F}$).

(a) ***Symmetry:*** *$Z(\sigma(\mathfrak{m})) = \sigma(Z(\mathfrak{m}))$, where σ acts on $\bigotimes_{i=1}^{n} Z(\Sigma_{g_i})$ by the map $v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$ and hence on its dual space by $\sigma(Z(\mathfrak{m})) = Z(\mathfrak{m}) \circ \sigma^{-1}$.*

(b) ***Nondegeneracy:*** *$Z(\hat{\mathfrak{d}}_g)$ is a nondegenerate pairing on $Z(\Sigma_g)$, which we'll call $\langle\ ,\ \rangle$. By symmetry it is a symmetric pairing, and let us define $Z(\hat{\mathfrak{e}}_g)$ to be the canonical dual element $\sum_j v_j \otimes w_j \in Z(\Sigma_g) \otimes Z(\Sigma_g)$ such that $\sum_j v_j \langle w_j, x\rangle = \sum_j \langle x, v_j\rangle \otimes w_j = x$.*

(c) ***Sewing:*** *If $\mathfrak{m}\colon \bigcup_{i=1}^{m} \Sigma_{g_i}$ and $\mathfrak{n}\colon \bigcup_{j=1}^{n} \Sigma_{g'_j}$ with $g_m = g'_1$ then $Z(\mathfrak{m}) \otimes Z(\mathfrak{n}) \circ \Phi = Z(\mathfrak{m} \cup_s \mathfrak{n})$, where*

$$\Phi\colon \bigotimes_{i=1}^{m-1} Z(\Sigma_{g_i}) \otimes \bigotimes_{j=2}^{n} Z(\Sigma_{g_j}) \to \bigotimes_{i=1}^{m} Z(\Sigma_{g_i}) \otimes \bigotimes_{j=1}^{n} Z(\Sigma_{g_j})$$

is the canonical map which is $Z(\hat{\mathfrak{e}}_{g_m})$ as a map $\mathbb{F} \to Z(\Sigma_{g_m}) \otimes Z(\Sigma_{g_m})$ tensored with the identity on the other factors.

(d) ***Mending:*** *If $\mathfrak{m}\colon \bigcup_{i=1}^{n} \Sigma_{g_i}$ with $g_1 = g_2$ then $Z(\mathfrak{m}_m) = Z(\mathfrak{m}) \circ \Phi$ where*

$$\bigotimes_{i=3}^{n} Z(\Sigma_{g_i}) \to \bigotimes_{i=1}^{n} Z(\Sigma_{g_i})$$

is the canonical map which is $Z(\hat{\mathfrak{e}}_{g_1})$ tensored with the identity on the other factors.

Then Z determines a TQFT, unique up to equivalence, whose value on the chosen surfaces and basic morphisms is as given.

Proof. If $\Sigma = \bigcup_{i=1}^n \Sigma_{g_i}$, define $Z(\Sigma) = \bigotimes_{i=1}^n Z(\Sigma_{g_i})$.

For each permutation morphism $\sigma\colon \bigcup_{i=1}^n \Sigma_{g_i} \to \bigcup_{i=1}^n \Sigma_{g_{\sigma(i)}}$, define $Z(\sigma)\colon \bigotimes_{i=1}^n Z(\Sigma_{g_i}) \to \bigotimes_{i=1}^n Z(\Sigma_{g_{\sigma(i)}})$ to be the corresponding permutation map. Because the permutation groupoid embeds into $\mathfrak{BC}'$ we have $Z(\sigma_1) \circ Z(\sigma_2) = Z(\sigma_1 \circ \sigma_2)$ and because of the symmetry axiom we have $Z(\mathfrak{m})Z(\sigma) = Z(\mathfrak{m} \circ \sigma)$ when $\mathfrak{m}$ is basic.

If $\mathfrak{m}\colon \bigcup_{i=1}^m \Sigma_{g_i} \to \emptyset$ is not basic because it has an underlying manifold which is not connected, then there exists a permutation σ such that $\mathfrak{m} \circ \sigma = \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_n$ is a union of basic morphisms. Define $Z(\mathfrak{m}) = (Z(\mathfrak{m}_1) \otimes \cdots \otimes Z(\mathfrak{m}_n)) \circ Z(\sigma^{-1})$. The choice of σ was not unique, since we could have reordered the factors $\mathfrak{m}_i$, but it is easy to check that $Z(\mathfrak{m})$ is well-defined, and that Z on this extended collection of morphisms satisfies all the axioms in the statement of the theorem as well as $Z(\mathfrak{m} \cup \mathfrak{n}) = Z(\mathfrak{m}) \otimes Z(\mathfrak{n})$ and $Z(\mathfrak{m} \circ \sigma) = Z(\mathfrak{m}) \circ Z(\sigma)$.

Now if $\Sigma = \bigcup_{i=1}^n \Sigma_{g_i}$, define $\Sigma^\dagger = \bigcup_{i=1}^n \Sigma_{g_{n-i+1}}$ and

$$\hat{\mathfrak{d}}_\Sigma = (\Sigma \times I, \Sigma^\dagger \cup \Sigma, \emptyset, f)$$

where f sends Σ to $\Sigma \times \{0\}$ via the identity map and $\Sigma^\dagger$ to $\Sigma \times \{1\}$ via the obvious permutation composed with the union of a copy of the S map for each Σ_{g_i}. It is easy to check that $Z(\hat{\mathfrak{d}}_\Sigma)$ is the pairing on $Z(\Sigma)$ determined by the pairings $\hat{\mathfrak{d}}_i$ on each Σ_{g_i}. Define $\hat{\mathfrak{e}}_\Sigma$ so that $(\hat{\mathfrak{d}}_\Sigma \otimes \mathbf{1}_\Sigma)(\mathbf{1}_\Sigma \otimes \mathfrak{e}_\Sigma) = \mathbf{1}_\Sigma$ and $(\mathbf{1}_{\Sigma^\dagger} \otimes \hat{\mathfrak{d}}_\Sigma)(\hat{\mathfrak{e}}_\Sigma \otimes \mathbf{1}_{\Sigma^\dagger}) = \mathbf{1}_{\Sigma^\dagger}$ and define $Z(\hat{\mathfrak{e}}_\Sigma)$ to be the element of $Z(\Sigma) \otimes Z(\Sigma^\dagger)$ dual to the pairing.

If $\mathfrak{m}\colon \Gamma \cup \Sigma \to \emptyset$ and $\mathfrak{n}\colon \Sigma^\dagger \cup \Delta \to \emptyset$ we can construct $\mathfrak{m} \cup_\Sigma \mathfrak{n}\colon \Gamma \cup \Delta \to \emptyset$, which is $(\mathfrak{m} \cup \mathfrak{n}) \circ (\mathbf{1}_\Gamma \cup \hat{\mathfrak{e}}_\Sigma \cup \mathbf{1}_\Delta)$, by a sequence of sewings, mendings, and permutations. It then follows from the sewing, mending and symmetry axioms that

$$Z(\mathfrak{m} \cup_\Sigma \mathfrak{n}) = (Z(\mathfrak{m}) \otimes Z(\mathfrak{n})) \circ \Phi$$

where Φ is formed by tensoring the identity map with the map associated to $\hat{\mathfrak{e}}_\Sigma$.

Finally, if $\mathfrak{m}\colon \Sigma \to \Gamma$ is an arbitrary element of $\mathfrak{BC}'$, define $\hat{\mathfrak{m}}\colon \Gamma^\dagger \cup \Sigma \to \emptyset$ by

$$\hat{\mathfrak{m}} = \hat{\mathfrak{d}}_\Gamma \circ (\mathbf{1}_{\Gamma^\dagger} \cup \mathfrak{m})$$

noting that

$$\mathfrak{m} = (\mathbf{1}_\Gamma \cup \hat{\mathfrak{m}}) \circ (\hat{\mathfrak{e}}_\Gamma \cup \mathbf{1}_\Sigma)$$

and define

$$Z(\mathfrak{m}) = (\mathbf{1}_{Z(\Gamma)} \otimes Z(\hat{\mathfrak{m}})) \circ (Z(\hat{\mathfrak{e}}_\Gamma) \cup \mathbf{1}_{Z(\Sigma)}).$$

We have only to check that the conditions after the definition of a TQFT are satisfied.

1. If $\mathfrak{n}\colon \Sigma \to \Gamma$ and $\mathfrak{m}\colon \Gamma \to \Delta$ then $Z(\mathfrak{m}\circ\mathfrak{n}) = Z(\mathfrak{m})\circ Z(\mathfrak{n})$. To see this note

$$\begin{aligned} Z(\widehat{\mathfrak{m}\circ\mathfrak{n}}) &= Z(\hat{\mathfrak{m}}\cup_\Gamma \hat{\mathfrak{n}}) \\ &= (Z(\hat{\mathfrak{m}})\otimes Z(\hat{\mathfrak{n}}))\circ(\mathbf{1}_{Z(\Delta^\dagger)}\otimes Z(\hat{\mathfrak{e}}_\Gamma)\otimes \mathbf{1}_{Z(\Sigma)}) \\ &= (Z(\hat{\mathfrak{d}}_\Delta)\otimes Z(\hat{\mathfrak{d}}_\Gamma))\circ(\mathbf{1}_{Z(\Delta^\dagger)}\otimes Z(\mathfrak{m})\otimes\mathbf{1}_{Z(\Gamma^\dagger)}\otimes Z(\mathfrak{n}))\circ(\mathbf{1}_{Z(\Delta)}\otimes Z(\hat{\mathfrak{e}}_\Gamma)\otimes\mathbf{1}_{Z(\Sigma)}) \\ &= Z(\hat{\mathfrak{d}}_\Delta)\circ(\mathbf{1}_{Z(\Delta^\dagger)}\otimes(Z(\mathfrak{m})\circ Z(\mathfrak{n}))) \end{aligned}$$

   and thus

$$\begin{aligned} Z(\mathfrak{m}\circ\mathfrak{n}) &= (\mathbf{1}_{Z(\Delta)}\otimes Z(\widehat{\mathfrak{m}\circ\mathfrak{n}}))\circ(Z(\hat{\mathfrak{e}}_\Delta)\otimes\mathbf{1}_{Z(\Sigma)}) \\ &= Z(\mathfrak{m})\circ Z(\mathfrak{n}). \end{aligned}$$

2. $Z(\mathbf{1}_\Sigma) = \mathbf{1}_{Z(\Sigma)}$. By definition

$$\begin{aligned} Z(\mathbf{1}_\Sigma) &= (\mathbf{1}_{Z(\Sigma)}\otimes Z(\widehat{\mathbf{1}_\Sigma}))\circ(Z(\hat{\mathfrak{e}}_\Sigma)\otimes\mathbf{1}_{Z(\Sigma)}) \\ &= (\mathbf{1}_{Z(\Sigma)}\otimes Z(\hat{\mathfrak{d}}_\Sigma))\circ(Z(\hat{\mathfrak{e}}_\Sigma)\otimes\mathbf{1}_{Z(\Sigma)}) \\ &= \mathbf{1}_{Z(\Sigma)}. \end{aligned}$$

3. $Z(\Sigma\cup\Gamma) = Z(\Sigma)\otimes Z(\Gamma)$. This is by definition.
4. $Z(\mathfrak{m}\cup\mathfrak{n}) = Z(\mathfrak{m})\otimes Z(\mathfrak{n})$. Suppose that $\mathfrak{m}\colon \Sigma_1\to\Gamma_1$ and $\mathfrak{n}\colon\Sigma_2\to\Gamma_2$, so that

$$\widehat{\mathfrak{m}\cup\mathfrak{n}} = (\hat{\mathfrak{m}}\cup\hat{\mathfrak{n}})\circ(\sigma_{\Gamma_1^\dagger\cup\Sigma_1,\Gamma_2^\dagger}\cup\mathbf{1}_{\Sigma_2})$$

   and thus

$$\begin{aligned} Z(\mathfrak{m}\cup\mathfrak{n}) &= (\mathbf{1}_{Z(\Gamma_1)\otimes Z(\Gamma_2)}\otimes Z(\widehat{\mathfrak{m}\cup\mathfrak{n}}))\circ(Z(\hat{\mathfrak{e}}_{\Gamma_1\cup\Gamma_2})\otimes\mathbf{1}_{Z(\Sigma_1)}\otimes\mathbf{1}_{Z(\Sigma_2)}) \\ &= Z(\mathfrak{m})\otimes Z(\mathfrak{n}). \end{aligned}$$

5. $Z(\emptyset) = \mathbb{F}$. This is by definition.
6. $Z(\sigma_{\Sigma,\Gamma}) = \sigma_{Z(\Sigma),Z(\Gamma)}$. This is also by definition.

□

Remark 5. From the point of view of the category theoretic definition of TQFT, the distinction between sewing and mending may seem more semantic than real. In fact the mending property plays a distinct and crucial role. Any 2-framed or ordinary three-manifold invariant can be extended in a formal fashion to something which satisfies all the other axioms (possibly at the cost of having infinite-dimensional vector spaces). In fact Reshetikhin and Turaev's original construction of the cobordism invariant [RT91] demonstrated all the axioms but mending, relying on fairly general properties of the link invariant and the modularity. The demonstration of the mending axiom [Tur94, Saw96] uses

properties of the link invariant which have much more to do with the connection to conformal field theory. Sewing is also the axiom which the Hennings invariant and its nonsemisimple cousins fail.

3. Surgery

The previous section reduced the problem of finding a TQFT from one about 2-framed cobordisms essentially to a question about 2-framed manifolds with boundary. This is still quite difficult to work with, and some combinatorial presentation is necessary. The most convenient is surgery.

First let us pick a convenient choice of representative objects Σ_g. Specifically, let each Σ_g be a handlebody of genus g.

Recall that an unoriented framed link in S^3 determines a (compact, connected, simply-connected, oriented) four-manifold with boundary as follows: Identify S^3 with the boundary of the four-ball B^4, thicken the components of the link to a collection of embedded tori in S^3 with a choice of longitude (the framing), and attach a two-handle to B^4 along each component of the link (the attaching part of the boundary of the two-handle has a preferred longitude, so the attachment map is determined up to isotopy). If the original S^3 contains an embedding of handlebodies and the link is chosen so as not to intersect the range of the embeddings, an embedding of the same handlebodies is determined in the boundary of the four-manifold. Of course we may view the four-manifold as a 2-framed three-manifold, and view the embedded handlebodies as its boundary. Thus a framed link together with a collection of nonintersecting embeddings of handlebodies Σ_g into the complement determines a basic 2-framed cobordism.

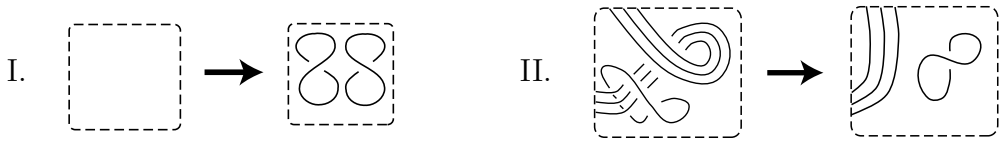

Figure 3. The framed Kirby moves with embedded handlebodies

Theorem 2. *Every basic 2-framed cobordism is diffeomorphic to one constructed by surgery on S^3 with an embedding of handlebodies. Two different pairs of embeddings and surgery link in S^3 give the same 2-framed cobordism if and only if they can be connected by isotopy and a sequence of 2-framed Kirby moves as shown in Figure 3, where the open strands pictured in the second move can be either strands of the surgery link or handles of the embedded handlebodies (here the handlebodies are*

assumed to come equipped with a choice of longitude for each handle, so that a projection of an embedding can be presented by a projection of the longitudes, a twist representing a Dehn twist as is the convention with framed links).

Proof. Of course two basic 2-framed cobordisms are the same if and only if there is they have the same signature and there is a diffeomorphism of the boundaries of the underlying four-manifold intertwining the embeddings of each Σ_g.

in [Rob97] Roberts proves that given two compact connected three-manifolds with boundary and a diffeomorphism from the boundary of one to the boundary of the other, there is a framed link in the interior of one such that surgery on that link gives a three-manifold with boundary over which the diffeomorphism extends to a diffeomorphism of the entire manifolds. Further, two such links can be connected by a sequence of moves O_1, O_2, O_3 shown in Figure 4 together with their mirror images and inverses, where the (respectively) ball, two-handled torus and torus can be embedded anywhere in the manifold.

We note first that in the proof of this result, move O_1 is used only to make the signatures of the linking matrices of the two links equal, after which every use of it can be a replaced with a use of move O_3, and thus the 2-framed version of Roberts' result would simply drop move O_1.

Of course the situation in the present theorem is a special case of Roberts' result, where one of the three-manifolds is S^3 with several handlebodies removed, and the parameterizations of the boundary give the diffeomorphism. Thus since it is clear that the 2-framed Kirby moves do not change the 2-framed three-manifold with embedded handlebodies, we need only prove that moves O_2 and O_3 can be replaced by a sequence of 2-framed Kirby moves.

This relies on the following two observations, essentially due to Fenn and Rourke [FR79]. The first is that, given a link and embedded handlebodies in S^3, and given a choice of curves on the boundary of the handlebodies which generate the homology of the handlebody, a sequence of Kirby moves can replace the embedding with one in which all these curves bound disks in S^3 minus the embedded handlebodies. The second is that if K_1 and K_2 are two links plus embeddings in S^3 which can be connected by a sequence of 2-framed Kirby moves, and if T is a framed link in one of the embedded handlebodies, and K_1', K_2' are the result of embedding T in to S^3 via the embedding of the handlebody and then removing that handlebody from the list, then K_1' and K_2' can be connected by a sequence of 2-framed Kirby moves.

For the first observation, notice the embedding of the handlebodies gives an embedding of the boundary curves into S^3, and as framed links they admit a projection. It is well-known that for any projection of a framed link, one can, by flipping certain of the crossings, make it a projection of a link in which all components are unlinked zero framed unknots. So apply 2-framed Kirby move I to add sufficiently many ± 1 framed unknot, and for each crossing that needs to be flipped, apply move II as shown in Figure 5 to flip it, noting as indicated by the dotted lines in that figure that the move is meant to be applied to the handlebodies, not just the boundary curves. The result of these moves will have the desired property.

For the second observation, notice the result is manifestly true for a single 2-framed Kirby move, and thus for an arbitrary sequence.

Figure 6 decomposes move O_3 into a sequence of 2-framed Kirby moves assuming that the torus is embedded so that the meridian plus the longitude bounds a disk in S^3 (here the vertical strands represent any number of link components which might go through the bounded disks). For an arbitrary embedding, use the two observations to show that a sequence of 2-framed Kirby moves will replace any embedding with one whose meridian plus longitude bounds a disk, apply Figure 6, then invert the sequence of Kirby moves to return to the original embedding.

The two observations again reduce the general problem of move O_2 to one where the two-handled torus is embedded as shown in Figure 7, and the moves in the figure prove the result in that case. □

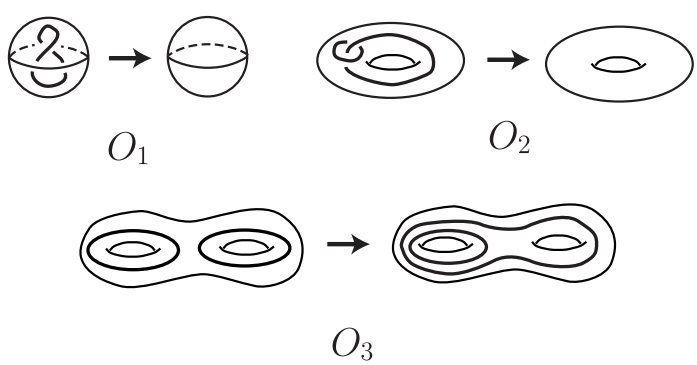

FIGURE 4. The Roberts moves

Remark 6. While [Saw96] construct TQFTs by proving invariance of appropriate quantities under the 2-framed Kirby moves, one could just as well use the Roberts moves directly.

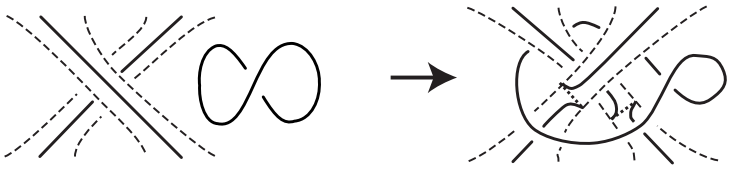

FIGURE 5. Flipping a crossing with move II

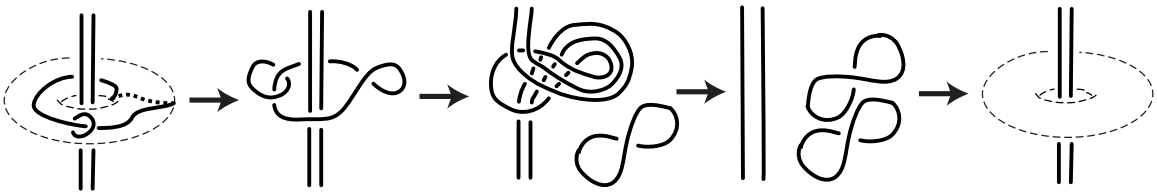

FIGURE 6. Enacting O_3 with 2-framed Kirby moves

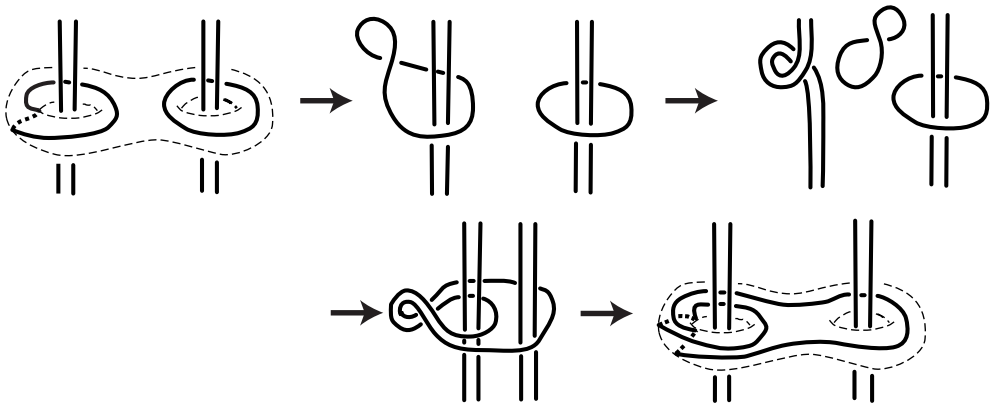

FIGURE 7. Enacting O_2 with 2-framed Kirby moves

Corollary 2. *For each g suppose Σ_g is a handlebody of genus g, in particular a 2-framed surface, and suppose $Z(\Sigma_g)$ is a finite-dimensional vector space. Given an embedding of $\bigcup_{i=1}^n \Sigma_{g_i}$ with each Σ_{g_i} labeled by a vector in $Z(\Sigma_{g_i})$ into S^3 together with a framed unoriented link in the complement of the embedding, suppose f is an invariant of the labeled embedding and the link. Suppose further that f is linear as a function of each label, independent of the ordering of $\{g_i\}_{i=1}^n$, unchanged by the framed Kirby moves and that the value of f on the embedding shown in Figure 8 (shown here for the genus two case) is a nondegenerate pairing $\langle v, w\rangle$ on $Z(\Sigma_g)$. Finally suppose that f of the embeddings pictured in*

Figure 9 are related by

$$f(A \cup_s B) = (f(A) \otimes f(B)) \circ \Phi$$
$$f(A_m) = f(A) \circ \Phi$$

where Φ is constructed out of the dual element to the pairing as in Theorem 1. Then f is actually an invariant of the basic cobordism determined by that embedding and link which satisfies the axioms of Theorem 1 and as such determines a TQFT.

Proof. Of course invariance under the biframed Kirby moves guarantees f is an invariant of the cobordism. We are given that it satisfies the Symmetry and Nondegeneracy Axioms of Theorem 1, and to see it satisfies Sewing and Mending it suffices to check that the embeddings and link shown in Figure 9 (shown here only for a genus two 2-framed surface, represented by its underlying graph) represent the sewing and mending of the basic cobordisms. This follow easily from the definition of the surgery description. □

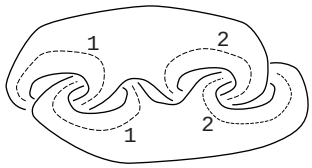

FIGURE 8. The embedding plus link corresponding to the pairing

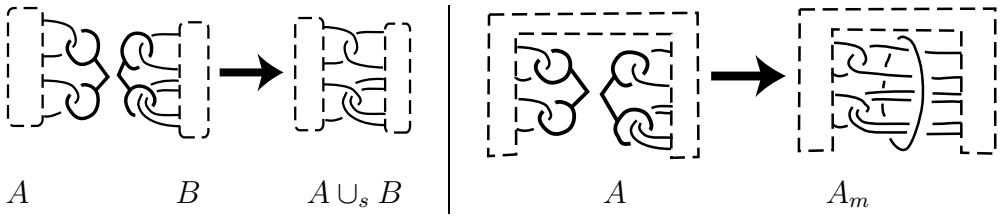

FIGURE 9. Sewing and mending of three-manifolds

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