2005

On retaining classical truths and classical deducibility in many-valued and fuzzy logics

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This is a post-print of an article published in *Journal of Philosophical Logic*. The original publication is available at http://www.springerlink.com

Peer Reviewed

Published Citation
DeWitt, Richard. 2005. On retaining classical truths and classical deducibility in many-valued and fuzzy logics. Journal of Philosophical Logic 34 (5-6), 545-560.

On Retaining Classical Truths and Classical Deducibility in Many-Valued and Fuzzy Logics

DeWitt, R. (2005) "On Retaining Classical Truths and Classical Deducibility in Many-Valued and Fuzzy Logics," *Journal of Philosophical Logic* 34; 545-560.

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Since the systematic study of many-valued logics began in earnest early in the 20th century, many-valued logics have received their fair share of scrutiny. One topic that arises often concerns the fact that the set of formulas validated by many-valued systems, and the inferences sanctioned by such systems, tend to differ substantially from those of classical logic (with some commentators viewing this as a virtue, some as a vice). One issue that does not seem to have been addressed, however, is the source of these differences between classical logic and many-valued logics.

My main interest in this paper is with this issue, that is, with the source of the differences between many-valued and classical logic with respect to the set of valid formulas and inferences sanctioned. In particular, I am interested in the question of whether such differences are inherent in many-valued systems. And if the differences are not inherent, then what is the source? The main focus of the current paper is on answering these two questions.

Many-valued logics are most often presented as semantic systems. I too take a semantic approach. I begin, in Section One, with a brief discussion of some of the differences between the set of formulas validated by classical semantics and those validated by the many-valued systems presented to date. I likewise discuss the differences between the inferences sanctioned by classical and many-valued semantics. In Section Two I turn to the question of whether the differences stem from the plurality of truth-values. The answer is negative: I show, in Section Two, that it is straightforward to construct a many-valued semantics validating all and only the classical truths, and sanctioning exactly the same inferences as classical semantics. Thus, whatever the source of the differences in the set of valid formulas and inferences sanctioned, they cannot stem simply from the plurality of truth-values. In Section Three I provide a brief discussion of the semantics presented in Section Two, and then in Section Four I turn to the second of my questions. In this section we identify the source of the differences, and Section Five then extends the results to take into account considerations unique to infinite-valued systems.

Section One: Classical and Many-Valued Truths and Inferences

There are a variety of ways to present many-valued semantics. For convenience, I will take the essential elements of a semantics to be (i) models consisting of specifications of domains, valuations on those domains, ranges of truth-values, and ranges of designated values, and (ii) a specification of truth-conditions.

Whereas in two-valued semantics a valid formula is one that always takes the value 1, there are, in many-valued semantics, some decisions to be made as to what shall count as valid. It is common to consider the "favored" truth-values, that is, those used to determine validity in the way that 1 is used in classical semantics, as *designated values*. The decision as to which values are to be considered designated is generally made by the author of the semantics, with the two most common choices being, on the one hand, just the value 1, and on the other hand, any value at least as great as .5.

The notion of designated values makes it straightforward to define many-valued counterparts of the usual semantic notions. A formula will be considered *valid* just in case it takes a designated value under every model, and a many-valued semantics will be said to *retain classical logic* just in case that semantic's valid formulas are exactly the classically valid formulas. A model *satisfies* a formula iff that formula receives a designated value under the model, and a model *simultaneously satisfies* a set of formulas ' just in case, under that model, every member of the set receives a designated value. We shall say that a set of formulas ' *implies* a formula A under a model M just in case M simultaneously satisfies ' and satisfies A, and that ' *semantically implies* A iff ' implies A under every model. Finally, a semantics S will be said to *retain the classical theory of deducibility* just in case whenever a set of formulas ' semantically implies a formula A under classical semantics, then ' semantically implies A under S, and vice versa.

Which values are taken as designated will, of course, affect which formulas are valid and which formulas are semantically implied by which sets of formulas (or, loosely speaking, which inferences are sanctioned by the semantics). These issues–the valid formulas (or lack thereof) and inferences sanctioned (or not sanctioned) by the various many-valued semantics–are central to the issue in which I am interested. The many-valued systems presented to date differ from classical logic in the set of valid formulas, the inferences sanctioned, or both.

Consider first the case in which 1 is taken as the only designated value. Then some of the better-known many-valued semantics–for example, Bochvar's internal system, Kleene's weak system, Kleene's strong system, and others–fail to validate even P 6 P. In general, where 1 is taken as the only designated value, none of the many-valued semantics studied to date retain classical logic.[1]

As a brief aside, the failure to retain classical logic is not, of course, necessarily a problem. These systems are often motivated by a feeling that not all of the classically valid formulas should be retained. Considerations about future contingents, vague predicates, quantum mechanics, and so on, have led some to the view that certain classically valid formulas–for example, excluded middle–ought not to be retained. But such systems tend to lose even formulas such as P 6 P. And it seems difficult to justify, on any interpretation of '6', a failure to validate P

6 P. At the very least, the loss of such formulas is a noteworthy difference between these systems and classical logic.

The above discussion assumed that the designated values included only the value 1. To see the other side of the coin, consider the other common choice for the range of designated values, in which all values greater than or equal to .5 are taken as designated. In this scenario, a number of many-valued systems will validate the full range of classical truths. Such systems, however, typically fail to retain the classical theory of deducibility. Again, it is not necessarily a problem that such many-valued systems fail to sanction every classically-acceptable inference. But it is noteworthy that such systems fail to respect inferences that seemingly ought to be sanctioned. For example, consider the semantic analog of *modus ponens*, in which ' = {P 6 Q, P}. Classically, of course, ' semantically implies Q, and intuitively, this seems right, even in many-valued contexts. But again, in many of the better-known systems, including Bochvar's internal system, Kleene's strong system, Kleene's weak system, Slupecki's (1946) system, and a range of others, ' no longer semantically implies Q. In general, where values other than 1 are taken as designated, the many-valued systems studied to date fail to retain the classical theory of deducibility. And again, this is a noteworthy difference between such systems and classical logic.

As noted, my main interest is in investigating whether such differences are inherent in many-valued systems, and if not, then what is the source of such differences. In the next section, we look at whether it is possible to construct a many-valued system that retains both classical logic and the classical theory of deducibility. The idea, of course, is that if a many-valued system can be constructed that retains both, then clearly the sorts of differences discussed in this section cannot be inherent in many-valued semantics.

Section Two: An Alternative Many-Valued Semantics

Let *MV semantics* consist of the set of MV models, where an *MV model* is a four-tuple +D,n,Tru,Des,. In such a model, D is a non-empty domain and n a valuation on that domain, subject to the conditions described below. Tru is the set of truth-values for the model. We require that Tru be finite and, as is usual, that {0,1} f Tru f [0,1]. (The reason for requiring that Tru be finite is discussed in Section Five.) Des is the set of designated truth-values for the model, which we specify to be those truth-values at least as great as .5, that is, Des = {x * x 0 Tru and x $ .5}.

As is generally the case with many-valued semantics, the truth-conditions for the connectives are *normal*, that is, they agree with the classical assignments whenever the truth-values are restricted to 0 and 1. A typical first-order language is presumed, the language having an infinite set T of terms, an infinite stock of variables, a two-place predicate symbol =, for each n $ 0 a (possibly empty) set P_n of n-place predicate letters, and symbols v, &, 6, ~, », ¼,), and (.

Conditions on Valuations: In an MV model +D,n,Tru,Des,, the function n is subject to the following conditions:

(i) for each d 0 D, d = n(t) for some t 0 T

(ii) for each t 0 T, n(t) 0 D

(iii) for each P 0 P_0, n(P) 0 Tru

(iv) for each P 0 P_n (n $ 1), n(P) 0 {R * R : D^n 6 Tru}

Truth Conditions: The truth-value *A*$_m$ of a formula A under an MV model M is given by the following clauses:

(i) for s, t 0 T, *s = t*$_m$ = 1 if n(s) = n(t); *s = t*$_m$ = 0 otherwise

(ii) for P 0 P_0, *P*$_m$ = n(P)

(iii) for P 0 P_n and t_1,...,t_n 0 T, *Pt_1...t_n*$_m$ = n(P)(n(t_1),...,n(t_n))

(iv) *~A*$_m$ = 1 if *A*$_m$ < .5; *~A*$_m$ = 0 otherwise

(v) *A v B*$_m$ = max{*A*$_m$, *B*$_m$}

(vi) *A & B*$_m$ = min{*A*$_m$, *B*$_m$}

(vii) *A 6 B*$_m$ = min{1, 1 - (*A*$_m$ - *B*$_m$)} if *B*$_m$ $ .5 or *A*$_m$ < .5;

A 6 B$_m$ = *B*$_m$ otherwise

(viii) *¼xA[x]*$_m$ = min{*A[t]*$_m$ * t 0 T}

(ix) *»xA[x]*$_m$ = max{*A[t]*$_m$ * t 0 T}

Where a formula A receives a designated value under an MV model M, we write ö$_m$ A; where A receives a designated value under every member of a set S of MV models, we write ö$_s$ A; where A is valid under MV semantics, that is, A receives a designated value under every MV model, we write ö$_{mv}$ A; and where A is classically valid we write ö$_c$ A. Likewise, where a set of formulas ' implies A under an MV model M, we write ' ö$_m$ A; where ' semantically implies A under MV semantics, we write ' ö$_{mv}$ A; and where ' semantically implies A under classical semantics, we write ' ö$_c$ A.

Lemma: Where TV is the set of MV models for which Tru = {0,1}, $\vDash_c A \iff \vDash_{tv} A$.

Proof: Since the truth-conditions for MV semantics are normal, it follows immediately that $\vDash_c A \iff \vDash_{tv} A$.

Proposition 1: $\vDash_c A \iff \vDash_{mv} A$.

Proof: Suppose $\vDash_c A$ and, for *reductio*, that $\nvDash_{mv} A$. Then for some MV model M = $\langle D,\nu,Tru,Des\rangle$, $\nvDash_m A$. Then $|A|_m < .5$. Let $M^* = \langle D,\nu^*,\{0,1\},\{1\}\rangle$ be an MV model where ν^* is such that (i) for each $t \in T$, $\nu^*(t) = \nu(t)$, (ii) for each $P \in P_0$, $\nu^*(P) = 1$ if $\nu(P) \geq .5$, and $\nu^*(P) = 0$ otherwise, and (iii) for each $P \in P_n$ and $t_1,\ldots,t_n \in T$, $\nu^*(P) = R$, where $R{:}D^n \to \{0,1\}$ is such that $R(\nu^*(t_1),\ldots,\nu^*(t_n)) = 1$ if $\nu(P)(\nu(t_1),\ldots,\nu(t_n)) \geq .5$, and $R(\nu^*(t_1),\ldots,\nu^*(t_n)) = 0$ otherwise. A straightforward induction on the complexity of formulas shows that $|A|_{m*} = 0 \iff |A|_m < .5$. So $|A|_{m*} = 0$, that is, $\nvDash_{m*} A$. Since $M^* \in TV$, it follows from the lemma that $\nvDash_c A$; contradiction. So $\vDash_c A \Rightarrow \vDash_{mv} A$.

On the other hand, suppose $\vDash_{mv} A$ and, for *reductio*, that $\nvDash_c A$. From the lemma it follows that $\nvDash_{tv} A$, from which it follows that $\nvDash_{mv} A$; contradiction. So $\vDash_{mv} A \Rightarrow \vDash_c A$.

Hence $\vDash_c A \iff \vDash_{mv} A$.

Proposition 2: $' \vDash_c A \iff ' \vDash_{mv} A$.

Proof: The proof proceeds, in all essential respects, like that of Proposition 1.

Propositions 1 and 2 show that MV semantics retains both classical logic and the classical theory of deducibility. This appears to answer our first question of whether the sorts of differences discussed in Section One are inherent in many-valued systems. Since MV semantics is a many-valued semantics without these differences, such differences cannot be inherent in many-valued semantics.

Section Three: Discussion

A number of issues call for discussion, the first of which concerns the truth-conditions found in the semantics of Section Two. Although I am not particularly concerned, in the current context, with arguing for or against various interpretations of the connectives, and hence with arguing over truth-conditions, nonetheless some comments on the subject are in order.

First, it is worth noting that given the truth-conditions for MV semantics, there is a sense in which the set of designated and undesignated truth-values are playing the roles played, respectively, by 1 and 0 in classical two-valued semantics. This comes through most clearly in the truth-conditions for negation, but also to an extent in the truth-conditions for the conditional. In addition, as shown above, MV semantics and classical semantics validate exactly the same formulas and sanction exactly the same inferences. Given these facts, it might be objected that MV semantics is not a genuine many-valued semantics, but rather, is more a two-valued semantics in disguise.

In response to this consideration, in a many-valued context it is important to keep in mind the notably different roles played by the designated and undesignated values, on the one hand, and the truth-values on the other. To help illustrate these roles, consider an analogy. With respect to grading student work, we can separate grades into two sets, passing grades and non-passing grades. The distinction between passing and non-passing grades plays certain important roles, for example, in determining who gets credit for courses, who is categorized as a first year student, a sophomore, junior or senior, who has earned the right to graduate, and so on. Within the broader sets of passing and non-passing grades, there are, of course, the individual grades, which play their own different, yet equally important roles. These roles include determining class rankings, who graduates with honors, who continues to receive scholarships, who is slated for academic probation, and so on. And although we separate these individual grades into two sets, the passing and the non-passing, we would never be tempted to view such a grading system as consisting of only two grades.

Likewise, in a many-valued semantics, the designated/undesignated values and the individual truth-values play different roles. The designated/undesignated values are used in determining which formulas are valid, whether a model simultaneously satisfies a set of formulas, whether a set of formulas semantically implies a particular sentence, and so on. On the other hand, in a many-valued context, the idea is to have the individual truth-values play other, equally important, roles. Depending on the particular semantics, and the intentions of the author of the semantics, such roles might include enabling the semantics to reflect that robins are more paradigmatic members of the class of birds than are penguins, that my 20-year-old single nephew is a more central example of a bachelor than is the Pope, that the sentence "grass is green" is true to a greater degree than is "the color of the cover of Hardin's *Color for Philosophers* is green," and so on. In short, the designated/undesignated values play an important role distinct from the role played by the individual truth-values. And just as we would never be led to view a typical grading system as really consisting of only two grades, there is no reason to consider a system such as MV semantics to be two-valued.

Similar considerations hold with respect to the fact that MV semantics and classical semantics validate exactly the same formulas and sanction exactly the same inferences. The formulas validated, and the inferences sanctioned, are certainly two important characteristics of a system. But they are not the only important characteristics. As indicated above, other characteristics–for example, whether a semantics can reflect facts such as that some individuals are members of a class to a greater degree than are other individuals–are also important. In fact, with respect to many-valued systems, these latter characteristics are often viewed as the more central characteristics. So although MV semantics and classical semantics share some characteristics, they fail to share other important characteristics, and thus MV semantics is not merely classical semantics in disguise.

This, then, firms up our answer from the end of Section Two: MV semantics is a many-valued semantics not having the sorts of differences discussed in the first section, and so clearly such differences cannot be inherent in many-valued systems. This leads us to our next main question, concerning the source of these differences.

Section Four: The Source of the Differences

In many-valued systems, intuitions concerning the appropriate truth-conditions for disjunction and conjunction are the most widely agreed on. In particular, there is general agreement that a disjunction should take the maximum value of the disjuncts, while a conjunction should take the minimum value of the conjuncts.

For various reasons, some of which are discussed more below, the quantifiers are somewhat more problematic. But in general, the following condition on universally quantified formulas is intuitively appealing:

¼xA[x]$_m$ 0 Des if *A[t]*$_m$ 0 Des for all t 0 T

¼xA[x]$_m$ ē Des otherwise,

while similar intuitions suggest the following condition on existentially quantified formulas:

»xA[x]$_m$ 0 Des if *A[t]*$_m$ 0 Des for some t 0 T

»xA[x]$_m$ ē Des otherwise.

In finitely-valued systems, universally quantified formulas generally take the minimum value of the formulas gotten by substituting terms for the variable in question, and existentially quantified formulas generally take the maximum value of such substitutions. As should be clear, such systems respect the conditions above. Such is not the case for typical infinite-valued systems, in which universally quantified formulas generally take the glb, and existentially quantified formulas take the lub, of the formulas gotten by substituting terms for the variable in question. Such systems do not respect the above conditions, and this is a point to which I will return in Section Five.

For the sake of convenience, let us call the above conditions on disjunction, conjunction, and the quantifiers the *standard conditions*. Again, the intuitions underlying these conditions are widespread, and even in the few systems (e.g., infinite-valued systems) that do not respect all of them, it is more likely that the failure is due to the fact that such systems cannot, in any natural way, respect the conditions (again, more on this in Section Five) rather than any disagreement with the intuitions.

Where we find the greatest variability in many-valued intuitions is in the truth-conditions for negation and for the conditional. As such, a more thorough discussion of these connectives is in order.

In many-valued semantics, two forms of negation are typically distinguished, these being choice negation and exclusion negation (sometimes referred to as "internal/external" or "word/sentence" negation). Consider a sentence such as 'the shirt is green.' The idea behind choice negation is that the negation operator "attaches" to the predicate in question, such that the sentence is best interpreted as 'the shirt is not-green.' Such a sentence is generally considered true to the degree that the object is a member of the anti-extension of the predicate in question. For example, if the shirt in question is a member of the class "green" to degree .8, then 'the shirt is not green' is true to degree .2. In general, many-valued systems with the following truth-condition for negation can be considered to be employing choice negation:

$*{\sim}A*_m = 1 - *A*_m$

In contrast, in exclusion negation, the negation operator ranges over the entire sentence, so that the sentence would properly be read as 'not (the shirt is green)'. Often, in systems employing exclusion negation, such a sentence is considered absolutely true if the object is not a member of the extension of the predicate, and absolutely false if the object is a member of the extension of the predicate. More generally, we can consider many-valued systems that respect the following condition to be employing exclusion negation:

$*{\sim}A*_m$ 0 Des if $*A*_m$ ē Des;

$*{\sim}A*_m$ ē Des otherwise.

(It is worth mentioning that choice and exclusion negation are not necessarily mutually exclusive. For example, an infinite-valued semantics in which Tru = [0,1] - {.5} could employ choice negation and still respect the condition on exclusion negation. A four-valued system in which Tru = {0, .25, .75, 1} could do likewise. However, in typical many-valued semantics, and in particular, any many-valued semantics whose range of truth-conditions include .5, choice negation and exclusion negation will be mutually exclusive.)

As mentioned, I am not here particularly interested in arguing for or against various interpretations of the connectives. However, I do want to make one brief point about choice and exclusion negation. It is clear that both forms of negation are found in ordinary discourse, and a many-valued semantics interested in reflecting (at least some of) the nuances of natural language ought to reflect this fact. It is worth noting that in the development of a typical semantics, the negation operator is usually presented as a sentential operator. Interpreted as such, the negation operator would most naturally be viewed as ranging over the sentence in question, and this speaks in favor of interpreting the negation operator in terms of exclusion negation.

Choice negation, on the other hand, acts on the predicate in question. As such, choice negation acts more like a predicate modifier, in the same way that 'almost' acts as a predicate modifier in the sentence 'Sara is almost tall'. Lakoff (1973) has proposed a straightforward and natural way of incorporating predicate modifiers into many-valued semantics. Using 'almost' as an example, the idea is to let this modifier "broaden" the extension of the predicate in question. If 'Sara is tall' is true, say, to degree .6, the effect of the modifier is that 'Sara is almost tall' comes out true to some appropriately greater degree, say .8. As mentioned, Lakoff has shown how straightforwardly to incorporate this idea into a many-valued semantics. Since choice negation behaves as a predicate modifier, the natural treatment would be along the lines suggested by Lakoff. The idea would be to let choice negation in effect flip the extension of the predicate in question, the effect being that if 'Sara is tall' is true to degree .6, then 'Sara is not tall' (where 'not' is treated in its choice sense and as a predicate modifier) is true to degree .4.

To summarize this idea, it seems most natural to interpret the negation operator as exclusion negation, as is done in MV semantics (and a number of other many-valued systems as well). Then, if the author of a semantics wishes to extend it to reflect more of the nuances of natural language, choice negation can straightforwardly be incorporated as a predicate modifier.

Let me turn now to the interpretation of the conditional in many-valued semantics. The debate over the "correct" interpretation of the conditional goes back at least to the ancient

Greeks, the principal players in the debate being Philo, on the one hand, and Diodorus on the other[2]. Philo argued, in essence, for the most "generous" interpretation of the conditional. In particular, a conditional should be true in any case where the antecedent is false or the consequent is true. The most natural way to extend this generous interpretation of the conditional to many-valued cases would be to say that a conditional should receive a designated value in any case where the antecedent is undesignated or the consequent designated. Along these lines, say that a semantics employs a *Philonian conditional* if the truth-condition for the conditional meets the following requirement:

A 6 B$_m$ 0 Des if *A*$_m$ ē Des or *B*$_m$ 0 Des

A 6 B$_m$ ē Des otherwise.

As is probably clear, MV semantics employs a Philonian conditional. Although some other previously-presented many-valued semantics do as well, the majority of many-valued systems employ a non-Philonian conditional.

Now, with these points about exclusion/choice negation and the Philonian conditional in place, we are in a position to present the following proposition.

Proposition 3: Let S be any semantics that respects the standard conditions for disjunction, conjunction, and the quantifiers. Then employing exclusion negation and a Philonian conditional are individually necessary and jointly sufficient for S to retain both classical logic and the classical theory of deducibility.

Proof:

Suppose S does not employ exclusion negation. There are two cases to consider.

Case (i): *A*$_m$ ē Des and *~A*$_m$ ē Des. Then *A v ~A*$_m$ ē Des, so ǿA v ~A.

Case (ii): $|A|_m \in$ Des and $|{\sim}A|_m \in$ Des. Then $|A \ \& \sim A|_m \in$ Des. Let B be any formula such that $|B|_m \notin$ Des. Then $A \ \& \sim A \nvDash_s B$.

So employing exclusion negation is necessary for S to retain both classical logic and the classical theory of deducibility.

Next, suppose S does not employ a Philonian conditional. There are three cases.

Case (i): $|A|_m \notin$ Des and $|A \to B|_m \notin$ Des. Then $|(A \to B) \vee A|_m \notin$ Des, so $\nvDash_s (A \to B) \vee A$.

Case (ii): $|B|_m \in$ Des and $|A \to B|_m \notin$ Des. Then $B \nvDash_s A \to B$.

Case (iii): $|A|_m \in$ Des, $|B|_m \notin$ Des, and $|A \to B|_m \in$ Des. Then $A \to B, A \nvDash_s B$.

So employing a Philonian conditional is likewise necessary for S to retain both classical logic and the classical theory of deducibility.

Finally, suppose S does employ both exclusion negation and a Philonian conditional. The proof that this is sufficient to retain both classical logic and the classical theory of deducibility proceeds much like that of Proposition 1. First, note that a similar lemma will hold. In particular, where S^* is the subset of models for which Tru = {0,1}, $\vDash_c A \Leftrightarrow \vDash_{s^*} A$. Now, suppose $\vDash_c A$ and, for *reductio*, that $\nvDash_s A$. Then for some model $M \in S$, $|A|_m \notin$ Des. Specify a two-valued model $M^* = \langle D, \nu^*, \{0,1\}, \{1\}$, where ν^* is such that (i) for each $t \in T$, $\nu^*(t) = \nu(t)$, (ii) for each $P \in P_0$, $\nu^*(P) = 1$ if $\nu(P) \in$ Des, and $\nu^*(P) = 0$ otherwise, and (iii) for each $P \in P_n$ and $t_1,\ldots,t_n \in T$, $\nu^*(P) = R$, where $R{:}D^n \to \{0,1\}$ is such that $R(\nu^*(t_1),\ldots,\nu^*(t_n)) = 1$ if $\nu(P)(\nu(t_1),\ldots,\nu(t_n)) \in$ Des, and $R(\nu^*(t_1),\ldots,\nu^*(t_n)) = 0$ otherwise. A straightforward induction will show that $|A|_{m^*} = 0 \Leftrightarrow |A|_m \notin$ Des, so $\nvDash_{s^*} A$ and thus, from the lemma, $\nvDash_c A$; contradiction. So $\vDash_c A \Rightarrow \vDash_s A$. On the other hand, suppose $\vDash_s A$ and, for *reductio*, that ö̸$_c$A. From the lemma it follows that ö̸$_{s*}$ A, and thus ö̸$_s$ A; contradiction. So

ö$_s$ A Y ö$_c$ A, and hence ö$_s$ A] ö$_c$ A. A similar proof shows that ' ö$_c$ A] ' ö$_s$ A.

Thus, employing exclusion negation and a Philonian conditional are individually necessary and jointly sufficient for S to retain both classical logic and the classical theory of deducibility.

Proposition 3, then, firms up the answer to the other main question of this paper, namely, the question as to the source of the differences discussed in Section One. As Proposition 3 shows, any semantics that fails to employ either exclusion negation or a Philonian conditional cannot retain both classical logic and the classical theory of deducibility. In contrast, any semantics that respects the standard conditions on disjunction, conjunction, and the quantifiers, and that employs exclusion negation and a Philonian conditional, will retain both classical logic and the classical theory of deducibility.

Section Five: Fuzzy Logics

With the above discussions in place, some issues concerning fuzzy logics can be clarified. Fuzzy logics, as with typical many-valued logics, are generally presented as semantic systems, albeit semantic systems with infinitely-many truth values. Typically, in infinite-valued systems, Tru = [0,1]. Such semantics would include earlier infinite-valued systems, such as that of

Lukasiewicz and Tarski (1930), or the more recent infinite-valued semantics based on fuzzy set theory, for example, those discussed in Zadeh (1975).

We know from Proposition 3 that if a semantics, including an infinite-valued semantics, respects the standard conditions for conjunction and disjunction, and employs exclusion negation and a Philonian conditional, then any failure of that semantics to retain both classical logic and the classical theory of deducibility will not be due to the truth conditions for these connectives.

What of the quantifiers? In most infinite-valued systems studied to date, including the systems mentioned above, the truth-conditions for the quantifiers are as follows:

¼xA[x]$_{m}$ = glb{*A[t]*$_{m}$ * t 0 T}

»xA[x]$_{m}$ = lub{*A[t]*$_{m}$ * t 0 T}

Call these the *typical quantifier conditions for infinite-valued semantics.*

Proposition 4: Let IV be any infinite-valued semantics that respects the standard conditions for conjunction and disjunction, and that employs exclusion negation and a Philonian conditional. If IV employs the typical quantifier conditions for infinite-valued semantics, then IV can retain neither classical logic nor the classical theory of deducibility.

Proof: Let Des = [n,1] (n > 0). (The proof is easily modified for the case where Des = (n,1]). Let P be a one-place predicate, and M = +D,n,Tru,Des, be a model where n(P) = R, such that for each t_i 0 T, R(t_i) = max{0 , n - 1/i}. Then lub{*Pt_i*$_{m}$ * t_i 0 T} = n, so *»xPx*$_{m}$ = n, and thus *»xPx*$_{m}$ 0 Des. Note that for all t_i 0 T, *Pt_i*$_{m}$ ē Des. Since IV employs exclusion negation, we know that for all t_i 0 T, *~Pt_i*$_{m}$ 0 Des, thus for all

$t_i \in T$, $|{\sim}Pt_i|_m \geq n$. So $\text{glb}\{|{\sim}Pt_i|_m \mid t_i \in T\} \geq n$, and hence $|\forall x{\sim}Px|_m \geq n$. That is, $|\forall x{\sim}Px|_m \in Des$, and so $|{\sim}\forall x{\sim}Px|_m \notin Des$. So $\exists xPx \nvDash_{IV} {\sim}\forall x{\sim}Px$.

Likewise, since IV employs a Philonian conditional, $|\exists xPx \rightarrow {\sim}\forall x{\sim}Px|_m \notin Des$. Hence $\nvDash_{IV} \exists xPx \rightarrow {\sim}\forall x{\sim}Px$.

So IV retains neither classical logic nor the classical theory of deducibility.

In short, an infinite-valued semantics that employs the typical quantifier conditions for infinite-valued systems cannot retain both classical logic and the classical theory of deducibility.[3]

As a brief aside, and to make note of a point that does not seem to have been discussed in the literature, the failure of the typical infinite-valued semantics studied to date to respect the standard conditions on the quantifiers seems an intuitively unappealing feature of such systems. That is, it is difficult to justify truth-conditions that allow an existentially-quantified formula to receive a designated value even though every formula gotten by substituting terms for the variable in question receives an undesignated value. Likewise, it is equally difficult to justify having a universally quantified formula receive an undesignated value even though every formula gotten by substituting terms for the variable in question receives a designated value. This, incidentally, goes some way toward explaining why the semantics of Section Two required the set of truth-values to be finite–such a requirement was the most straightforward way of respecting the standard conditions on the quantifiers.

Proposition 4 shows that any infinite-valued semantics that employs the typical quantifier conditions for infinite-valued semantics cannot retain both classical logic and the classical theory of deducibility. However, it is perfectly straightforward to construct infinite-valued semantics that do retain both. To give one example among many, consider a system in which the truth-values consist of the reals in the interval [0,1], but excluding .5, that is, Tru = [0,1] - {.5}, and let Des = (.5,1].[4] The truth-values for the quantifiers can be similar to the typical quantifier conditions for infinite-valued semantics, but modified in some appropriate manner to handle the cases involving .5. The truth-conditions for the universal quantifier might be, for example,

$|\forall x A[x]|_m = .51$ if $\text{glb}\{|A[t]|_m \mid t \in T\} = .5$

$|\forall x A[x]|_m = \text{glb}\{|A[t]|_m \mid t \in T\}$ otherwise,

while the conditions for the existential quantifier might be

$|\exists x A[x]|_m = .49$ if $\text{lub}\{|A[t]|_m \mid t \in T\} = .5$

$|\exists x A[x]|_m = \text{lub}\{|A[t]|_m \mid t \in T\}$ otherwise.

(The values .51 and .49, as used in these truth-conditions, are of course somewhat arbitrary, and any number of other values would work as well.) It is easy to see that such a system will respect the standard conditions on the quantifiers, in spite of employing an infinity of truth-values. Thus, so long as such a system respects the standard conditions for disjunction and conjunction, and employs exclusion negation and a Philonian conditional, then Proposition 3 guarantees that the system will retain both classical logic and the classical theory of deducibility.

So the failure of the infinite-valued systems studied to date to retain both classical logic and the classical theory of deducibility should not be taken to suggest that such differences are inherent in infinite-valued systems. Rather, as Proposition 4 shows, the quantifier conditions typically used in previously-presented infinite-valued systems do not respect the standard conditions on the quantifiers. And this, at bottom (along perhaps with the truth-conditions for negation and the conditional, as shown in Proposition 3), is the reason the best-known infinite-valued systems do not retain classical logic and the classical theory of deducibility.

One final note: the above discussion shows that it is possible to have infinite-valued semantics that are axiomatizable, contrary to what has sometimes been suggested.[5] For example, any infinite-valued system that respects the standard conditions for conjunction, disjunction, and the quantifiers, and that employs exclusion negation and a Philonian conditional, will validate exactly the classically-valid formulas and so of course will be axiomatizable. Thus, we likewise see that the inability to axiomatize the typical infinite-valued semantics studied to date does not stem from the infinity of truth values employed.

Section Six: Conclusion

This, then, completes the answers to the main questions of this paper. The answer to our first question, whether differences between classical and many-valued semantics are inherent in many-valued systems, was negative. As we saw in Section Two, it is not difficult to specify a many-valued system that validates all and only the classically valid formulas while at the same time sanctioning exactly the same inferences as those sanctioned by classical logic.

Our second main question concerned the source of the differences. We saw in Section Four that if a many-valued system respects the standard conditions for disjunction, conjunction, and the quantifiers, then employing exclusion negation and a Philonian conditional are individually necessary and jointly sufficient for that semantics to retain both classical logic and the classical theory of deducibility. Thus for such a system (that is, one respecting the standard conditions on disjunction, conjunction, and the quantifiers), a failure to retain classical logic or the classical theory of deducibility stems from the truth conditions for negation and/or the conditional.

Although the results of Section Four hold for both finitely-valued and infinite-valued systems, some issues involving the better-known infinite-valued systems make it worthwhile to discuss such systems separately. In particular, as noted in Section Five, the infinite-valued systems studied to date do not respect the standard conditions for the quantifiers. As Proposition 4 shows, the quantifier conditions used in the best-known infinite-valued systems are themselves sufficient to prevent such systems from retaining either classical logic or the classical theory of deducibility, even if exclusion negation and a Philonian conditional are used.

However, we saw at the end of Section Five that it is not difficult to specify an infinite-valued system that does respect the standard conditions for the quantifiers. Thus, so long as such a system employs exclusion negation and a Philonian conditional (as well as the standard conditions for disjunction and conjunction), then the system will validate exactly the classically valid formulas while also retaining the classical theory of deducibility (and so, of course, any such infinite-valued system would also be axiomatizable).

REFERENCES

Kneale, W. and Kneale, M. (1962) *The Development of Logic*, Clarendon Press.

Lakoff, G. (1973) "Hedges: A Study in Meaning Criteria and the Logic of Fuzzy Concepts," *Journal of Philosophical Logic* 2, pp. 458-508.

Lukasiewicz, J. and Tarski, A. (1930) "Investigations into the Sentential Calculus," in Borkowski, L. (ed.) *Jan Lukasiewicz: Selected Works*, North-Holland Publishing Company.

Morgan, C. and Pelletier, F. (1977) "Some Notes Concerning Fuzzy Logics," *Linguistics and Philosophy* 1, pp. 79-97.

Rescher, N. (1969) *Many-Valued Logic*, McGraw-Hill.

Scarpellini, B. (1962) "Die Nicht-Axiomatisierbarkeit des Unednlichwertigen Pradikatenkalkuls von-Lukasiewicz," *Journal of Symbolic Logic* 27, pp. 159-170.

Slupecki, J. (1946) "The Complete Three-Valued Propositional Calculus," *Annales Universitatis Mariae Curie-Sklodawska*, vol. 1, pp. 193-209.

Zadeh, L. (1975) "Fuzzy Logic and Approximate Reasoning," *Synthese* 30, pp. 407-428.

ENDNOTES

I would like to thank George Schumm, Stewart Shapiro, and an anonymous referee for helpful comments on various ideas contained in, and versions of, this paper.

1. Details on many of the systems discussed in this section can be found in Rescher (1969).
2. See Kneale and Kneale (1962) for further elaboration.
3. My original notes for this paper were lost some years ago as a result of a fire. But if memory serves me correctly, the key idea in the proof of Proposition 4 of letting $R(t_i) = \max\{0 , n - 1/i\}$, or at least an idea similar to this, was first suggested to me by Stewart Shapiro.
4. A similar example is noted by Morgan and Pelletier (1977).
5. For example, Morgan and Pelletier (1977) claim this, citing Scarpellini (1962) as a source.