1-1-2002

Groups Whose Universal Theory Is Axiomatizable by Quasi-Identities

Benjamin Fine
Fairfield University, fine@fairfield.edu

Anthony M. Gaglione

A. G. Myasnikov

Dennis Spellman

Original published version can be found at DOI: 10.1515/jgth.2002.009

Peer Reviewed

J. Group Theory **5** (2002), 365–381

Journal of Group Theory

Groups whose universal theory is axiomatizable by quasi-identities

Benjamin Fine, Anthony M. Gaglione, Alexei Myasnikov and Dennis Spellman

(Communicated by D. J. S. Robinson)

Abstract. Discriminating groups were introduced in [3] with an eye toward applications to the universal theory of various groups. In [6] it was shown that if G is any discriminating group, then the universal theory of G coincides with that of its direct square $G \times G$. In this paper we explore groups G whose universal theory coincides with that of their direct square. These are called *square-like groups*. We show that the class of square-like groups is first-order axiomatizable and contains the class of discriminating groups as a proper subclass. Further we show that the class of discriminating groups is not first-order axiomatizable.

1 Introduction

A *discriminating group* is a group G such that every group separated by G is discriminated by G. Discriminating groups were introduced in [3] with an eye toward applications to the universal theory of various groups. In [6] various important examples of discriminating groups were given. These include Thompson's group F, the commutator subgroup of the Gupta–Sidki groups and some of Grigorchuk's groups of intermediate growth. Further in [6] it was shown that, for a finitely generated, equationally noetherian group G (see Section 2), the minimal universally axiomatizable class containing G coincides with the quasivariety generated by G if and only if G is discriminating. In that same paper it was shown that if G is any discriminating group, then the universal theory of G coincides with that of its direct square $G \times G$. In this paper we further explore groups G whose universal theory coincides with that of their direct square. A group G is termed *square-like* if the universal theory of G coincides with the universal theory of its direct square $G \times G$. Thus every discriminating group is square-like. We prove that the class of square-like groups is first-order axiomatizable. Recall that this means that this class of groups is the model class for some set of first-order sentences (see Section 2). Further we show that a group is square-like if and only if the minimal universally axiomatizable class containing G coincides with the quasivariety generated by G. Moreover we prove that the class of discriminating groups is a proper subclass of the class of square-like groups and further the class of discriminating groups is not first-order axiomatizable. To do this we give an example of a non-discriminating square-like group and an example of a discriminating group elementarily equivalent to a non-discriminating group.

In Section 2 we give the necessary preliminaries from both group theory and logic. In Section 3 we first review some necessary results on discriminating groups and then introduce square-like groups and prove that the class of discriminating groups is a proper subclass of the class of square-like groups and further that the class of discriminating groups is non-axiomatic. We then prove our main results on square-like groups. In Section 4 we consider square-like abelian groups and give sufficient conditions on an abelian group A so that A is square-like if and only if it is discriminating. Finally we close with some open questions.

2 Preliminaries in group theory and logic

We start with some necessary definitions and results from group theory.

Definition 1. Let $\mathcal{X}$ be a non-empty class of groups and let H be a group. Then $\mathcal{X}$ *separates* H provided that for every non-trivial element $h \in H$ there exist a group $G_h \in \mathcal{X}$ and a homomorphism $\varphi_h : H \to G_h$ such that $\varphi_h(h) \neq 1$. The class $\mathcal{X}$ *discriminates* H provided that for every finite non-empty set S of non-trivial elements of H there exist $G_S \in \mathcal{X}$ and a homomorphism $\varphi_S : H \to G_S$ such that $\varphi_S(h) \neq 1$ for all $h \in S$. If $\mathcal{X} = \{G\}$ consists of a single group then we say that G *separates* (*discriminates*) H. We say that $\mathcal{X}$ is a *separating family of groups* provided that every group G separated by $\mathcal{X}$ lies in $\mathcal{X}$.

Observe that a separating family of groups is closed under isomorphism. This is so since if $H \cong G \in \mathcal{X}$, then an isomorphism $\varphi : H \to G$ does not annihilate any non-trivial element of H; hence G separates H and so $H \in \mathcal{X}$.

Now let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite set of ordered distinct *variables*. For each positive integer n let $X_n = \{x_1, x_2, \ldots, x_n\}$ and let $F(X_n)$ be the free group with base X_n. Then a non-empty subset S of $F(X_n)$ shall be viewed as a system of equations

$$\{w(x_1, x_2, \ldots, x_n) = 1 : w \in S\}.$$

A *solution* of S in a group G shall be an ordered n-tuple $(g_1, \ldots, g_n) \in G^n$ such that

$$w(g_1, \ldots, g_n) = 1$$

in G for all $w \in S$.

Definition 2. A group G is *equationally noetherian* provided that for all positive integers n and all subsets $S \subseteq F(X_n)$ there is a finite subset $S_0 \subseteq S$ such that the systems S_0 and S have precisely the same solutions in G^n.

It was shown in [2] that non-abelian free groups and more generally linear groups over unital noetherian commutative rings are equationally noetherian. In particular all abelian groups are equationally noetherian.

We now introduce prevarieties. Our work on discriminating groups and square-like groups involves both prevarieties and quasivarieties. Quasivarieties will be introduced later in this section. Recall that a variety of groups can be characterized as a class of groups closed under subgroups, quotients and arbitrary cartesian products; see [10, p. 14].

Definition 3. A *prevariety* of groups is a class of groups $\mathscr{X}$ satisfying the following two properties:

(1) $\mathscr{X}$ is closed under subgroups;

(2) $\mathscr{X}$ is closed under cartesian products (of arbitrary indexed families $(G_i)_{i \in I}$ of groups from $\mathscr{X}$).

Observe that since the trivial group 1 is a subgroup of any group G, every prevariety $\mathscr{X}$ must contain at least 1. Note also that the intersection of any family of prevarieties is again a prevariety; so, if $\mathscr{Y}$ is any class of groups there is a least prevariety $\mathrm{pvar}(\mathscr{Y})$ containing $\mathscr{Y}$. This is the prevariety *generated* by $\mathscr{Y}$. In the case that $\mathscr{Y} = \{G\}$ is a singleton, we write $\mathrm{pvar}(G)$ for $\mathrm{pvar}(\mathscr{Y})$ and call $\mathrm{pvar}(G)$ the prevariety *generated by* G. The following theorem can be deduced from work of Birkhoff [1] from which group varieties can be classified as *closed classes* of groups (see [10] or [11]).

Theorem 1. *Let $\mathscr{X}$ be a class of groups. Then $\mathscr{X}$ is a prevariety of groups if and only if $\mathscr{X}$ is a separating family of groups.*

Proof. Suppose first that $\mathscr{X}$ is a prevariety of groups. Let the group H be separated by $\mathscr{X}$. For each $h \neq 1$ in H there is a group $G_h \in \mathscr{X}$ and a homomorphism $\varphi_h : H \to G_h$ such that $\varphi_h(h) \neq 1$. Then H embeds into $\prod_{h \in H - \{1\}} G_h$; hence H is, up to isomorphism, a subgroup of a cartesian product of groups in $\mathscr{X}$. It follows that H lies in $\mathscr{X}$ whenever H is separated by $\mathscr{X}$. We have thus proven that every prevariety is separating.

Now let $\mathscr{X}$ be a separating family of groups. Suppose that $H \in \mathscr{X}$ and G is a subgroup of H. Given any $g \neq 1$ in G, the inclusion map $\iota : G \to H$ does not annihilate g. Thus G is separated by $\mathscr{X}$; hence G lies in $\mathscr{X}$. Thus $\mathscr{X}$ is closed under subgroups. Now let $(G_i)_{i \in I}$ be an indexed family from $\mathscr{X}$. Let $g = (g_i) \neq 1$ lie in $\prod_{i \in I} G_i$. Choose $i_0 \in I$ such that $g_{i_0} \neq 1$. If $\pi : \prod_{i \in I} G_i \to G_{i_0}$ is projection onto the i_0-coordinate, then $\pi(g) = g_{i_0} \neq 1$. Hence $\mathscr{X}$ separates $\prod_{i \in I} G_i$; therefore $\prod_{i \in I} G_i$ lies in $\mathscr{X}$. Thus $\mathscr{X}$ is closed under cartesian products. We have shown that every separating family of groups is a prevariety of groups.

We now present the necessary preliminaries from model theory and logic. Let L be the first-order language with equality containing a constant symbol 1, a unary operation symbol $^{-1}$ and a binary operation symbol $\cdot$. We shall be considering only those L-structures which are groups; hence, here and in what follows, we (tacitly) assume the group axioms.

We remark that being *first-order* means that in the intended interpretation of any formula or sentence all of the variables (free or bound) are assumed to take on as values only individual group elements and never, for example, subsets of nor functions on the group in which they are interpreted.

If Φ is a consistent set of sentences of L, then the class $\mathbb{M}(\Phi)$ of all groups G satisfying every sentence φ in Φ is the *model class* of Φ. Note that every such class is non-empty and closed under isomorphism. If $\mathcal{X}$ is a class of groups, then $\mathcal{X}$ is *axiomatic* provided that there is at least one set Φ of sentences of L such that $\mathcal{X} = \mathbb{M}(\Phi)$.

Suppose that G and H are groups and $\lambda : G \to H$ is a function which preserves the truth of formulas in the following sense: for every integer $n \geqslant 0$ and every formula $\varphi(x_1, \dots, x_n)$ of L containing at most the distinct free variables $x_1, \dots, x_n$ it is the case that, for every ordered n-tuple $(g_1, \dots, g_n) \in G^n$, $\varphi(\lambda(g_1), \dots, \lambda(g_n))$ is true in H if and only if $\varphi(g_1, \dots, g_n)$ is true in G. We claim that such a map must be a group monomorphism. If $g_3 = g_1 g_2$ in G, then applying the above to the formula $x_3 = x_1 \cdot x_2$, we get $\lambda(g_3) = \lambda(g_1)\lambda(g_2)$; so that λ is homomorphic. Furthermore applying the above to the formula $x_1 = 1$, we conclude that $\lambda(g) = 1$ implies that $g = 1$, so that λ is, as claimed, monic.

Definition 4. Let G and H be groups and $\lambda : G \to H$ be a group monomorphism. Then λ is an *elementary embedding* provided that λ preserves the truth of formulas. If G is a subgroup of H and the inclusion map $\iota : G \to H$ is an elementary embedding, then we say that H is an *elementary extension of* G.

Examples. (1) Let G be a group and let I be a non-empty set. Let $\delta : G^I \to G$ be the diagonal map, i.e., $\delta(g)(i) = g$ for all $i \in I$. Let D be an ultrafilter on I. Then the map d from G into the *ultrapower* G^I/D given by $g \mapsto [\delta(g)]_D$ is an elementary embedding.

(2) Let $\lambda : G \to H$ be an isomorphism from the group G *onto* the group H. Then λ is an elementary embedding.

Since every sentence of L is a formula of L containing no free variables, we immediately deduce that the existence of an elementary embedding $\lambda : G \to H$ is a sufficient condition for G and H to satisfy precisely the same sentences of L.

Definition 5. Let G and H be groups. Then G and H are *elementarily equivalent* provided that they satisfy precisely the same sentences of L.

Example. Let F be a free group of countably infinite rank with basis $A = \{a_n : n < \omega\}$. Let F' be the commutator subgroup of F. Then F and F' are elementarily equivalent since they are isomorphic. However F is *not* an elementary extension of F'. For example, the formula

$$\exists x\,([[a_0, a_1], x] = [[a_0, a_1], a_2])$$

is true in F but false in F'. Here $[x, y]$ is the commutator $x^{-1}y^{-1}xy$.

We wish now to consider various special kinds of sentences of L. A *universal sentence* of L is one of the form

$$\forall \bar{x}\, \varphi(\bar{x}),$$

while an *existential sentence* is one of the form

$$\exists \bar{x}\, \varphi(\bar{x})$$

where $\bar{x}$ is a tuple of variables and $\varphi(\bar{x})$ is a quantifier-free formula of L where the only free variables are the variables in the tuple $\bar{x}$. (Vacuous quantifications are permitted and each of $\forall x\psi$ and $\exists x\psi$ is logically equivalent to ψ if the variable x does not occur in ψ.)

A universal sentence of the form

$$\forall \bar{x}\, \Big(\bigwedge_i (u_i(\bar{x}) = 1) \to (w(\bar{x}) = 1) \Big)$$

is called a *quasilaw* or *quasi-identity*. Note that every identity $\forall \bar{x}(w(\bar{x}) = 1)$ is equivalent to a quasilaw $\forall \bar{x},\ y\ ((y \cdot y^{-1} = 1) \to (w(\bar{x}) = 1))$.

Definition 6. Let $\mathscr{X}$ be a non-empty class of groups. The *universal closure* $\mathrm{ucl}(\mathscr{X})$ of $\mathscr{X}$ is the model class of the set of all universal sentences φ true in every group G in $\mathscr{X}$. A *quasivariety* is the model class of a set of quasilaws. The quasivariety $\mathrm{qvar}(\mathscr{X})$ *generated by* $\mathscr{X}$ is the model class of those quasilaws true in every group in $\mathscr{X}$. If $\mathscr{X} = \{G\}$ is a singleton, then we write $\mathrm{ucl}(G)$ for the universal closure of G and $\mathrm{qvar}(G)$ for the quasivariety generated by G.

The following facts are immediate.

(1) $\mathrm{ucl}(\mathscr{X})$ and $\mathrm{qvar}(\mathscr{X})$ are axiomatic. Moreover $\mathrm{ucl}(\mathscr{X})$ is the least universally axiomatizable class containing $\mathscr{X}$ and $\mathrm{qvar}(\mathscr{X})$ is the least quasivariety containing $\mathscr{X}$.

(2) The model class operator reverses inclusions; that is, if Φ and Ψ are consistent sets of sentences of L and $\Phi \subseteq \Psi$, then $\mathbb{M}(\Psi) \subseteq \mathbb{M}(\Phi)$. It follows from this that $\mathrm{ucl}(\mathscr{X}) \subseteq \mathrm{qvar}(\mathscr{X})$.

(3) It is straightforward to verify that every quasivariety contains the trivial group 1, is closed under subgroups and is closed under cartesian products. Thus every quasivariety is an axiomatic prevariety. We shall presently see that the converse is also true so that every axiomatic prevariety is a quasivariety. However the next example shows that not every prevariety need be axiomatic and hence need not be a quasivariety.

Example. Call an abelian group *reduced* provided that it contains no non-trivial divisible subgroup. It is straightforward to verify that the class of all reduced abelian groups is a prevariety. One can produce an ultrapower (see [5]) of $\mathbb{Z}$ which contains a copy of $\mathbb{Q}$. It follows that pvar($\mathbb{Z}$) is not axiomatic; hence pvar($\mathbb{Z}$) is not a quasivariety.

We now state (without proof) a series of lemmas (and consequences thereof) which are well known to model theorists. The reader may refer to [4, 5, 7] for more details.

Lemma 1. *Let $\mathscr{X}$ be a class of groups. Then $\mathscr{X}$ is axiomatic if and only if $\mathscr{X}$ is closed under both ultraproducts and elementary equivalence.*

Lemma 2. *Let G and H be groups. Then every universal sentence of L true in G is also true in H if and only if H is embeddable in an elementary extension ${}^*G$ of G.*

Lemma 3. *Let $\mathscr{X}$ be an axiomatic class of groups. Then $\mathscr{X}$ is universally axiomatizable* (*i.e., has at least one set of universal axioms*) *if and only if $\mathscr{X}$ is closed under subgroups.*

Lemma 4. *Reduced products preserve elementary equivalence* (*i.e., if I is a non-empty set, and G_i is elementarily equivalent to H_i for all $i \in I$ and D is a proper filter on I, then $\prod_{i \in I} G_i/D$ is elementarily equivalent to $\prod_{i \in I} H_i/D$*).

Corollary 1. *Elementary equivalence is preserved by cartesian products and ultraproducts in the sense of Lemma* 4.

Lemma 5. *Let $\mathscr{X}$ be an axiomatic class of groups. If $\mathscr{X}$ is closed under products of two factors, then $\mathscr{X}$ is closed under cartesian products of an arbitrary number* (*finite or infinite*) *of factors.*

Lemma 6. *Let $\mathscr{X}$ be a universally axiomatizable class of groups. Then $\mathscr{X}$ is a quasivariety if and only if the trivial group* 1 *lies in $\mathscr{X}$* (*equivalently, $\mathscr{X}$ is non-empty*) *and $\mathscr{X}$ is closed under cartesian products.*

Corollary 2 (Mal'cev). *A prevariety is a quasivariety if and only if it is axiomatic.*

Proof. We have already observed that a quasivariety is an axiomatic prevariety. Let $\mathscr{X}$ be an axiomatic prevariety. Certainly $\mathscr{X}$ is non-empty by Definition 3. Moreover, $\mathscr{X}$ is universally axiomatizable by Lemma 3. Since $\mathscr{X}$ is a prevariety, it is closed under cartesian products, again by Definition 3. Thus $\mathscr{X}$ is a quasivariety by Lemma 6.

In general, given two groups G and H, there are no known criteria (other than the definition) to determine whether or not G and H are elementarily equivalent. However, Szmielew [12] has completely characterized elementary equivalence of abelian groups. Recall that a group G has *finite exponent* if there is a positive integer n such that $x^n = 1$ for all $x \in G$ and G has *infinite exponent* otherwise. Szmielew distinguishes between two types of linear independence in an abelian group A (which we shall write additively). If m is a positive integer and $(a_i)_{i \in I}$ is a sequence of elements of A containing only finitely many non-zero terms, then $(a_i)_{i \in I}$ is *linearly independent modulo* m provided that

$$\sum_{i \in I} n_i a_i = 0 \Rightarrow n_i \equiv 0 \pmod{m}$$

for all $i \in I$.

The sequence $(a_i)_{i \in I}$ is *linearly independent modulo m in the stronger sense* provided that

$$\sum_{i \in I} n_i a_i \in mA \Rightarrow n_i \equiv 0 \pmod{m}$$

for all $i \in I$.

Szmielew then defines, for each prime p and each positive integer k, three quantities $\rho^{(i)}[p,k](A)$, $i = 1,2,3$, each of which is either a non-negative integer or the symbol ∞, as follows:

(1) $\rho^{(1)}[p,k](A)$ is the maximum number (if it exists) of elements of order p^k which are linearly independent modulo p^k;

(2) $\rho^{(2)}[p,k](A)$ is the maximum number (if it exists) of elements linearly independent modulo p^k in the stronger sense;

(3) $\rho^{(3)}[p,k](A)$ is the maximum number (if it exists) of elements of order p^k which are linearly independent modulo p^k in the stronger sense.

Theorem A (Szmielew [12]). *Let A and B be abelian groups. Then A and B are elementarily equivalent if and only if the following two conditions are satisfied*:

(1) *either A and B both have finite exponent or they both have infinite exponent*;

(2) *for all primes p and positive integers k, one has $\rho^{(i)}[p,k](A) = \rho^{(i)}[p,k](B)$ for $i = 1,2,3$.*

If G is a group, let $\mathrm{Th}_\forall(G)$ denote the set of all universal sentences of L true in G. (Note that $\mathrm{ucl}(G) = \mathbb{M}(\mathrm{Th}_\forall(G))$.) We shall say that two groups G and H are *universally equivalent* provided that $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(H)$, that is, they have the same universal theory. Since the negation of a universal sentence is logically equivalent to an existential sentence and vice-versa, two groups have the same universal theory if and only if they have the same existential theory. We may write the *matrix* of an existential formula in disjunctive normal form; that is, every existential sentence of L is logically equivalent to one of the form $\exists \bar{x} \left(\bigvee_i \varphi_i(\bar{x}) \right)$ where $\varphi_i(\bar{x})$ is a conjunction

$$\bigwedge_j (p_{ij}(\bar{x}) = 1) \wedge \bigwedge_k (q_{ik}(\bar{x}) \neq 1)$$

of equations and inequations. But $\exists\bar{x}\,(\bigvee_i \varphi_i(\bar{x}))$ is logically equivalent to $\bigvee_i \exists\bar{x}\,(\varphi_i(\bar{x}))$ and a disjunction will hold if and only if at least one of the disjuncts is true. Thus two groups G and H will be universally equivalent if and only if every finite system

$$p_i(\bar{x}) = 1 \quad (1 \leqslant i \leqslant I)$$
$$q_j(\bar{x}) \neq 1 \quad (1 \leqslant j \leqslant J)$$

of equations and inequations in finitely many variables $\bar{x} = (x_1, \dots, x_n)$ has a solution in G if and only if it has a solution in H.

3 Square-like groups

In this section we introduce square-like groups, proving that the class of discriminating groups is a proper subclass of the class of square-like groups and further that the class of discriminating groups is non-axiomatic. We then prove that the class of square-like groups is axiomatic and that a group is square-like if and only if it is universally equivalent to a discriminating group. Before introducing square-like groups we review some of the material on discriminating groups. Baumslag, Myasnikov and Remeslennikov proved the following result.

Theorem B ([2]). *The group G is discriminating if and only if G discriminates $G \times G$.*

It follows then that G being isomorphic to $G \times G$ is a sufficient condition for G to be discriminating. From Theorem B the following can be deduced.

Theorem C ([6]). *If G is discriminating, then G and $G \times G$ have the same universal theory.*

Motivated by this theorem we define the notion of a square-like group.

Definition 7. A group G is termed *square-like* if G and $G \times G$ have the same universal theory, that is, if $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(G \times G)$.

It follows from Theorem C that every discriminating group is square-like. Further, since G embeds in $G \times G$, every universal sentence true in $G \times G$ is automatically true in G. Thus a necessary and sufficient condition for a group G to be square-like is that every universal sentence true in G must also be true in $G \times G$. In [6] the following was proved, which tied together the notions of discrimination, being equationally noetherian, universal closure and quasivariety.

Theorem D ([6]). *Let G be a finitely generated, equationally noetherian group. Then G is discriminating if and only if* $\mathrm{ucl}(G) = \mathrm{qvar}(G)$.

We now give our first main result.

Theorem 2. *The class of discriminating groups is a proper subclass of the class of square-like groups. Further the class of discriminating groups is non-axiomatic.*

Proof. From Theorem C it follows that the class of discriminating groups is contained in the class of square-like groups. To complete the proof we first present an example of a square-like group which is not a discriminating group. We then give an example of a discriminating group that is elementarily equivalent to a non-discriminating group. From this second example it follows that the class of discriminating groups is not axiomatic. To construct these examples we first need the following lemma which is of interest in its own right.

Lemma 7. *The class of discriminating groups is closed under forming reduced products but not under direct unions.*

Proof. Let I be a non-empty set and let $(G_i)_{i\in I}$ be a family of discriminating groups indexed by I. Suppose D is a proper filter on I and let $\prod_{i\in I} G_i/D$ be a reduced product (see [5] or [7]). We must show that $\prod_{i\in I} G_i/D$ discriminates $(\prod_{i\in I} G_i/D) \times (\prod_{i\in I} G_i/D)$. It suffices to show that $\prod_{i\in I} G_i/D$ discriminates $\prod_{i\in I}(G_i \times G_i)/D$. Let $[(f_1, g_1)]_D, \dots, [(f_n, g_n)]_D$ be finitely many non-trivial elements of $\prod_{i\in I}(G_i \times G_i)/D$. Then for all j with $1 \leqslant j \leqslant n$,

$$\{i \in I : (f_j(i), g_j(i)) = 1\} \notin D.$$

For each $i \in I$, let $J(i) = \{j : (f_j(i), g_j(i)) \neq 1\}$. If $J(i)$ is empty, let $\varphi_i : G_i \times G_i \to G_i$ be projection onto the first coordinate. Otherwise, choose $\varphi_i : G_i \times G_i \to G_i$ such that $\varphi_i(f_j(i), g_j(i)) \neq 1$ for all $j \in J(i)$. This is possible since each G_i is discriminating. We then get an induced map

$$^*\varphi : \prod_{i\in I}(G_i \times G_i)/D \to \prod_{i\in I} G_i/D.$$

We claim that $^*\varphi$ does not annihilate $[(f_j, g_j)]_D$ for $j = 1, 2, \dots, n$. Suppose that this is false. Fix j and write $[(f, g)]_D$ for $[(f_j, g_j)]_D$. If $^*\varphi$ annihilates $[(f, g)]_D$, then

$$\{i \in I : \varphi_i(f(i), g(i)) = 1\} \in D.$$

But we claim that

$$\{i \in I : \varphi_i(f(i), g(i)) = 1\}$$

coincides with

$$\{i \in I : (f(i), g(i)) = 1\}.$$

For if $(f(i), g(i)) = 1$, then certainly $\varphi_i(f(i), g(i)) = 1$, while, on the other hand, if $(f(i), g(i)) \neq 1$ then $j \in J(i)$ and $\varphi_i(f(i), g(i)) \neq 1$ by the choice of φ_i. But then

$$\{i \in I : \varphi_i(f(i), g(i)) = 1\} = \{i \in I : (f(i), g(i)) = 1\}$$

does not lie in D; so $\varphi([(f, g)]_D) \neq 1$, contrary to our assumption. Thus $\prod_{i \in I} G_i/D$ discriminates $\prod_{i \in I}(G_i \times G_i)/D$ and $\prod_{i \in I} G_i/D$ is discriminating whenever G_i is discriminating for all $i \in I$. In other words, the class of discriminating groups is, as claimed, closed under reduced products. (In particular, it is closed under cartesian products.). We must now show that it is not closed under direct unions. To do this we need the following theorem of Baumslag, Myasnikov and Remeslennikov.

Theorem E ([2]). *Let A be a torsion abelian group. Suppose that for each prime p, the p-primary component of A modulo its maximal divisible subgroup contains no non-trivial element of infinite p-height. Then A is discriminating if and only if the following two conditions are satisfied for each prime p:*

(1) *for every positive integer k, $\rho^{(1)}[p, k](A)$ is either 0 or ∞;*

(2) *the rank of the maximal divisible subgroup of the p-primary component of A is either zero or infinite.*

Here the *rank* of a divisible abelian p-group is the maximal number of direct summands isomorphic to the quasicyclic group $\mathbb{Z}_{p^\infty}$; moreover, the p-height of an element a of an abelian p-group A is (with respect to A) the maximal positive integer n, if it exists, such that the equation $p^n x = a$ has a solution in A.

Now for each positive integer k let M_k be a free module of countably infinite rank over the ring $(\mathbb{Z}/2^k\mathbb{Z})$. Let M be the direct sum of the abelian groups M_k as k varies over the positive integers. Let $D = \mathbb{Z}_{2^\infty}$ be a rank 1 divisible 2-group. Let A be the direct sum $M \oplus D$. Then $A/D \cong M$ is a torsion abelian 2-group containing no non-trivial elements of infinite 2-height. By Theorem E, A is not discriminating since its maximal divisible subgroup D has rank 1. (It clearly suffices to restrict ourselves to the prime 2 since, if p is an odd prime, the p-primary component of A is 0.) But A is the direct union of the family $A_k = M \oplus (\mathbb{Z}/2^k\mathbb{Z})$ of subgroups as k varies over the positive integers. Each A_k is discriminating, since clearly $A_k \cong M$ for all k, and $M \cong M \times M$ is discriminating. Thus the class of discriminating groups is not closed under direct unions.

From this last proof we can complete part of the proof of Theorem 2. By Szmielew's Theorem (Theorem A), A and M as given above are elementarily equivalent since they both have infinite exponent and, if $p = 2$, then $\rho^{(i)}[p, k](A) = \rho^{(i)}[p, k](M) = \infty$ for $i = 1, 2, 3$ and for all k and, if $p \neq 2$, then $\rho^{(i)}[p, k](A) = \rho^{(i)}$, $[p, k](M) = 0$ for $i = 1, 2, 3$ and for all k. Thus the discriminating group M is elementarily equivalent to the non-discriminating group A. It follows that the class of discriminating goups is not axiomatic.

To show that the class of discriminating groups is proper in the class of square-like groups we need the following result.

Lemma 8. *The class of square-like groups is closed under reduced products and direct unions.*

Proof. Let I be a non-empty set and let $(G_i)_{i \in I}$ be a family of square-like groups indexed by I. Let D be a proper filter on I. For each $i \in I$, every universal sentence true in G_i is also true in $G_i \times G_i$. Hence, by Lemma 2, $G_i \times G_i$ embeds in an elementary extension ${}^*G_i$ of G_i. That induces an embedding $\prod_{i \in I}(G_i \times G_i)/D \hookrightarrow \prod_{i \in I} {}^*G_i/D$. Thus every universal sentence true in $\prod_{i \in I} {}^*G_i/D$ must also be true in $\prod_{i \in I}(G_i \times G_i)/D$. But $\prod_{i \in I} {}^*G_i/D$ is elementarily equivalent to $\prod_{i \in I} G_i/D$. Hence every universal sentence true in $\prod_{i \in I} G_i/D$ must also be true in $\prod_{i \in I}(G_i \times G_i)/D$. But $\prod_{i \in I} (G_i \times G_i)/D$ is isomorphic to $(\prod_{i \in I} G_i/D) \times (\prod_{i \in I} G_i/D)$. Thus every universal sentence true in $\prod_{i \in I} G_i/D$ must also be true in its direct square. It follows that the reduced product $\prod_{i \in I} G_i/D$ is square-like whenever each G_i is square-like. Therefore the class of square-like groups is closed under reduced products.

Now suppose that G is the direct union of a family $\mathscr{F}$ of square-like groups. Let $\mathscr{G}$ be the set of all $H \times H$ with $H \in \mathscr{F}$. Clearly $G \times G$ is the direct union of the family $\mathscr{G}$ of subgroups. Now let φ be a universal sentence of L true in G. Then φ must be true in every subgroup $H \in \mathscr{F}$. But each such H is square-like; hence φ is true in $H \times H$ for all $H \times H \in \mathscr{G}$. Universal sentences are easily seen to be preserved in direct unions. Therefore φ is true in $G \times G$. Thus every universal sentence true in G must also be true in $G \times G$, i.e., the class of square-like groups is closed under direct unions. It follows also that the class of square-like groups is closed under cartesian products and ultraproducts.

From this we can give the example of a square-like group which is not discriminating. Consider the non-discriminating group $A = M \oplus D$ given in the example prior to Lemma 8. This group was the direct union of the family $A_k = M \oplus (\mathbb{Z}/2^k\mathbb{Z})$ of discriminating (hence square-like) subgroups and therefore is itself square-like. This completes the proof of Theorem 2.

In contrast to the class of discriminating groups our next result shows that the class of square-like groups is indeed axiomatic.

Theorem 3. *The class of square-like groups is axiomatic.*

Proof. In view of Lemma 1 it will suffice to show that the class of square-like groups is closed under ultraproducts and elementary equivalence. However Lemma 8 showed that the class of square-like groups is closed not only under ultraproducts but even under arbitrary reduced products. Thus it will suffice to show that the class of square-like groups is closed under elementary equivalence.

So suppose that G is square-like and H is elementarily equivalent to G. Then, by Corollary 1, $H \times H$ is elementarily equivalent to $G \times G$. In particular,

$$\mathrm{Th}_\forall(H) = \mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(G \times G) = \mathrm{Th}_\forall(H \times H),$$

so that H is also square-like.

We note that although universal sentences are involved in the definition of square-like groups, this class does not have a set of universal axioms. If it did it would be closed under subgroups. Let V be a vector space of countably infinite dimension over the two element field. Then V (viewed as an abelian group) is discriminating (hence square-like) since $V \cong V \times V$. The subgroups of V of order 2 are not square-like since, for example, the universal sentence $\forall x, y, z\ ((x = y) \vee (x = z) \vee (y = z))$ is true in $\mathbb{Z}/2\mathbb{Z}$ but false in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. (A similar argument shows that no non-trivial finite group can be square-like.) However, since the class of square-like groups is closed under direct unions, it does have, by a theorem of Łoś and Susko [9], a set of so-called *universal–existential* axioms.

Although the previous two theorems distinguish the class of square-like groups from its subclass of discriminating groups, the next result shows that they coincide in the presence of a finite presentation.

Theorem 4. *Let G be a finitely presented group. Then G is discriminating if and only if it is square-like.*

Proof. If G is discriminating, it is square-like by Theorem C. Now we suppose that G is square-like and we must show that it is discriminating.

Let

$$G = \langle x_1, \dots, x_n; R_1, \dots, R_m \rangle$$

be a finite presentation for G where $R_i = R_i(x_1, \dots, x_n)$ are words in $x_1, \dots, x_n$. To show that G is discriminating we must show that G discriminates $G \times G$.

A finite presentation for $G \times G$ is then given by

$$\begin{aligned} G \times G = \langle x_1, \dots, x_n, y_1, \dots, y_n; R_1(x_1, \dots, x_n) = 1, \dots, R_m(x_1, \dots, x_n) = 1, \\ R_1(y_1, \dots, y_n) = 1, \dots, R_m(y_1, \dots, y_n) = 1, [x_i, y_j] = 1, i, j = 1, \dots, n \rangle. \end{aligned}$$

Now suppose that $W_1, \dots, W_k$ are non-trivial elements of $G \times G$. Then each W_i is given by a word $W_i(x_1, \dots, x_n, y_1, \dots, y_n)$ in the generators of $G \times G$. Consider now the existential sentence

$$\begin{aligned} \exists x_1, \dots, x_n, y_1, \dots, y_n \Bigg(\Bigg(\bigwedge_{i=1}^{m} R_i(x_1, \dots, x_n) = 1 \Bigg) \wedge \Bigg(\bigwedge_{i=1}^{m} R_i(y_1, \dots, y_n) = 1 \Bigg) \\ \wedge \Bigg(\bigwedge_{i,j} [x_i, y_j] = 1 \Bigg) \wedge \Bigg(\bigwedge_{i=1}^{k} W_i(x_1, \dots, x_n, y_1, \dots, y_n) \neq 1 \Bigg) \Bigg). \end{aligned}$$

This existential sentence is clearly true in $G \times G$. Since G is square-like, G and $G \times G$ have the same universal theory. Hence they have the same existential theory and therefore the above existential sentence is true in G. Therefore there exists elements $a_1, \dots, a_n, b_1, \dots, b_n$ in G such that $R_i(a_1, \dots, a_n) = 1$ for $i = 1, \dots, m$; $R_i(b_1, \dots, b_n) = 1$ for $i = 1, \dots, m$; $[a_i, b_j] = 1$ for $i, j = 1, \dots, n$ and $W_i(a_1, \dots, a_n, b_1, \dots, b_n) \neq 1$ for $i = 1, \dots, k$. The map from $G \times G$ to G given by mapping x_i to a_i and y_i to b_i for $i = 1, \dots, n$ defines a homomorphism for which the images of $W_1, \dots, W_k$ are nontrivial. Hence G discriminates $G \times G$ and therefore G is discriminating.

An argument similar to the proof of Theorem 3 shows that the class of all groups H for which there exists a discriminating group G_H elementarily equivalent to H is axiomatic. Clearly, this class is the least axiomatic class containing the discriminating groups. We now show that being square-like is equivalent to being universally equivalent to a discriminating group.

Theorem 5 (Main Theorem). *Let G be a group. Then the following three conditions are equivalent*:

(1) *G is square-like*;

(2) $\mathrm{ucl}(G) = \mathrm{qvar}(G)$;

(3) *G is universally equivalent to a discriminating group*.

Proof. (1) $\Rightarrow$ (2) Assume that G is square-like. Since $1 \leqslant G$, every universal sentence true in G must be true also in the trivial group 1. Hence $1 \in \mathrm{ucl}(G)$. Now let $H, K \in \mathrm{ucl}(G)$. Then by Lemma 2 there are elementary extensions G_H and G_K of G such that H is embeddable in G_H and K is embeddable in G_K. Then $H \times K$ is embeddable in $G_H \times G_K$. By Corollary 1, $G_H \times G_K$ is elementarily equivalent to $G \times G$. Hence $\mathrm{Th}_\forall(G_H \times G_K) = \mathrm{Th}_\forall(G \times G)$. But $\mathrm{Th}_\forall(G \times G) = \mathrm{Th}_\forall(G)$ since G is square-like. Hence every universal sentence true in G must also be true in $H \times K$. It follows that $\mathrm{ucl}(G)$ is closed under direct products of two factors. By Lemma 5, $\mathrm{ucl}(G)$ is closed under arbitrary cartesian products. Then, by Lemma 6, $\mathrm{ucl}(G)$ is a quasivariety, which must clearly coincide with the least quasivariety $\mathrm{qvar}(G)$ containing G.

(2) $\Rightarrow$ (3) Assume that $\mathrm{ucl}(G) = \mathrm{qvar}(G)$. Let I be an infinite index set. Since $G \in \mathrm{qvar}(G)$, we must have $G^I \in \mathrm{qvar}(G) = \mathrm{ucl}(G)$. Hence every universal sentence true in G must also be true in G^I. But every universal sentence true in G^I must also be true in G since G embeds in G^I, It follows that $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(G^I)$. Finally note that G^I is discriminating since $G^I \times G^I$ is isomorphic to G^I.

(3) $\Rightarrow$ (1) Suppose that H is a discriminating group and $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(H)$. Since H is discriminating it is square-like and thus $\mathrm{ucl}(H) = \mathrm{qvar}(H)$ by the first implication of this proof. Then $G \in \mathrm{ucl}(H) = \mathrm{qvar}(H)$ implies that

$$G \times G \in \mathrm{qvar}(H) = \mathrm{ucl}(H) = \mathrm{ucl}(G).$$

Hence every universal sentence true in G must also be true in $G \times G$ and so G is square-like.

Corollary 3. *Let G be a finitely generated, equationally noetherian group. Then G is square-like if and only if it is discriminating.*

4 Abelian groups

We give sufficient conditions for an abelian group to be square-like if and only if it is discriminating. Throughout this section we assume that our groups are abelian. We start with the following:

Definition 8. An abelian group A is *universally standardizable* or US provided that for each prime p and positive integer k, $\rho^{(1)}[p,k](A)$ is either zero or infinite.

Next we need some terminology from [8, p. 26].

Definition 9 ([8]). The *skeleton* of a group G is the class of all finitely generated groups that can be embedded in G.

Lemma 9. *Let A be a US-group.*

(1) *If $\rho^{(1)}[p,k](A) = \infty$, then for all integers n with $1 \leqslant n \leqslant k$,*

$$\rho^{(1)}[p,n](A) = \infty.$$

(2) *If $\rho^{(1)}[p,k](A) = 0$, then for all integers n with $n \geqslant k$,*

$$\rho^{(1)}[p,n](A) = 0.$$

Proof. (1) Suppose that $(a_1, a_2, \ldots)$ is an infinite sequence of elements of A of order p^k which are linearly independent modulo p^k. Let n be an integer with $1 \leqslant n \leqslant k$. Then we claim that the sequence $(p^{k-n}a_1, p^{k-n}a_2, \ldots)$ is linearly independent modulo p^n; so, in particular, its terms are distinct. (Note that $p^{k-n}a_i$ has order p^n since a_i has order p^k.) Suppose that this is false. Then there are integers $n_1, n_2, \ldots, n_N$, not all divisible by p^n, such that $n_1 p^{k-n}a_1 + \cdots + n_N p^{k-n}a_N = 0$. But this contradicts the assumption that $(a_1, a_2, \ldots)$ are linearly independent modulo p^k because not all of the integers $n_1 p^{k-n}, \ldots, n_N p^{k-n}$ are divisible by p^k. The contradiction shows, as claimed, that the sequence $(p^{k-n}a_1, p^{k-n}a_2, \ldots)$ is linearly independent modulo p^n. Assertion (1) follows. We note that (2) follows from (1) since A is US.

We note that if an abelian group A is not finitely generated, then its torsion subgoup T is not necessarily a direct summand in A; moreover, T is the direct sum $\bigoplus_p T_p$ where T_p is the p-primary component of A.

Definition 10. Let A be a US-group. Then its *universal standardization* is the abelian group $S_\forall(A)$ constructed as the direct sum $T \oplus F$ of a torsion abelian group T and a free abelian group F as follows: $F = 0$ if A contains no elements of infinite order;

otherwise, F is free abelian of rank the smaller of r and $\aleph_0$ where

$$r = \max\{\mathrm{rank}(A_0) : A_0 \text{ is free abelian and } A_0 \leqslant A\}.$$

For each prime p let A_p be the p-primary component of A and let T_p be the p-primary component of T. Let $T_p = 0$ if $A_p = 0$. Assume that $A_p \neq 0$. If A_p has finite exponent p^n, then T_p is the direct sum of a countable infinity of copies of the cyclic group $\mathbb{Z}/p^n\mathbb{Z}$ of order p^n; otherwise, T_p is the direct sum of a countable infinity of copies of the quasicyclic group $\mathbb{Z}_{p^\infty}$.

Theorem F ([2]). *Let A be an abelian group with torsion subgroup T.*

(1) *If A is discriminating, then T is discriminating.*

(2) *If T is a direct summand in A, then A is discriminating if and only if T is discriminating.*

Theorem 6. *Let A be an abelian group. Then A is square-like if and only if A is US.*

Proof. Suppose first that A is not US. Then for some prime p and positive integer k we have

$$0 < \rho^{(1)}[p,k](A) = n < \infty.$$

Then A satisfies the following universal sentence but $A \times A$ does not:

$$\forall x_1, \dots, x_n, x_{n+1} \left(\bigvee_{\bar{m} \neq 0} \left(\sum_{i=1}^{n+1} m_i x_i = 0 \right) \right).$$

Here $\bar{m} = (m_1, \dots, m_{n+1})$ varies over all $(n+1)$-tuples of integers m_i where $0 \leqslant m_i < p^k$ for $i = 1, 2, \dots, n+1$, with the exception of the zero vector $\mathbf{0} = (0, 0, \dots, 0)$. It follows that if A is not US, then A is not square-like. Now suppose that A is US. By the construction of $S_\forall(A)$ the groups A and $S_\forall(A)$ have the same skeleton. But a finite system of equations and inequations

$$\sum_{k=1}^{n} c_{ik} x_k = 0 \quad (1 \leqslant i \leqslant I)$$

$$\sum_{k=1}^{n} d_{jk} x_k \neq 0 \quad (1 \leqslant j \leqslant J)$$

(in finitely many variables) will have a solution in an abelian group B if and only if it has a solution in some finitely generated subgroup B_0 of B. It follows that A and $S_\forall(A)$ have the same universal theory. Now T is a direct summand in $S_\forall(A)$ and T is discriminating since $T \cong T \times T$ by the very construction of $S_\forall(A)$. It follows from Part (2) of Theorem F that $S_\forall(A)$ is discriminating. Thus if A is US, then A is universally equivalent to a discriminating group. But then A is square-like by Theorem 5. Hence A is square-like if and only if A is US.

Corollary 4. *Let A be an abelian group with torsion subgroup T. Then A is square-like if and only if T is square-like.*

Proof. This is obvious since clearly, for all primes p and positive integers k, $\rho^{(1)}[p,k](A) = \rho^{(1)}[p,k](T)$.

Corollary 5. *Let A be a torsion abelian group. For each prime p let A_p be the p-primary component of A. Suppose that for all primes p, A_p has finite exponent. Then A is square-like if and only if A is discriminating.*

Proof. Since every discriminating group is square-like we assume that A is square-like and show that it is discriminating. First of all, for each prime p, the p-primary component A_p of A has maximal divisible subgroup 0. This is so since if A_p contains even a single copy of the quasicyclic group $\mathbb{Z}_{p^\infty}$ then A_p will have infinite exponent, contrary to hypothesis. Hence A_p modulo its maximal divisible subgroup is just A_p itself. Now assume, to deduce a contradiction, that $a \neq 0$ is an element of A_p having infinite p-height. Let the exponent of A_p be p^N. Then the equation $p^N x = a$ must have a solution (say $x = b$) in A_p. But $b \in A_p$ and $p^N b = a$ and $p^N b = a \neq 0$ contradicts the assumption that A_p has exponent p^N. The contradiction shows, as claimed, that A_p contains no non-trivial element of infinite p-height. We are now in a position to apply the criteria of Theorem E. Since A is square-like, Theorem 6 above shows that, for each prime p and positive integer k, $\rho^{(1)}[p,k](A)$ is either 0 or ∞. Moreover we have already observed that the rank of the maximal divisible subgroup of A_p is zero for all primes p. hence by Theorem E, A is discriminating.

5 Open questions

(1) Is every square-like group a direct union of discriminating groups?

(2) Is the class of square-like groups the least axiomatic class containing the discriminating groups? Equivalently, must every square-like group be elementarily equivalent to a discriminating group?

(3) Is every finitely generated square-like group discriminating?

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Received 18 June, 2001; revised 7 December, 2001

B. Fine, Department of Mathematics, Fairfield University, Fairfield, Connecticut 06430, U.S.A.

A. M. Gaglione, Department of Mathematics, United States Naval Academy, Annapolis, Maryland 21402, U.S.A.

A. G. Myasnikov, Department of Mathematics, City College of CUNY, New York, New York 10031, U.S.A.

D. Spellman, Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19132, U.S.A.