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On Lyndon's equation in some Λ -free groups and HNN extensions

L. Ciobanu, B. Fine and G. Rosenberger

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Abstract. In this paper we study Lyndon's equation $x^p y^q z^r = 1$, with x, y, z group elements and p, q, r positive integers, in HNN extensions of free and fully residually free groups, and draw some conclusions about its behavior in Λ -free groups.

1 Introduction

The classical result of Lyndon and Schützenberger ([9]) states that any elements x, y and z of a free group F that satisfy the relation $x^p y^q = z^r$ for $p, q, r \geq 2$ necessarily commute. In the paper of Brady, Ciobanu, Martino and O'Rourke ([1]) this result has been generalized to Λ -free groups. In particular, the following result has been obtained. Let G be a group that acts freely on a Λ -tree, where Λ is an ordered abelian group, and let x, y, z be elements in G . If $x^p y^q = z^r$ with integers $p, q, r \geq 4$, then x, y and z must commute. It has been unclear whether the same conclusion holds for p, q, r not all larger than 4, and in particular the proof in [1] cannot be extended to these smaller integer cases. Here we shed light on the behavior of this equation in some HNN extensions and show that for p, q, r not all larger than 4 the conclusion of [1] does not always hold (see Corollary 1). This work complements the results in [5], where Lyndon's equation is studied in various amalgams of groups.

2 Results

Theorem 1. *Let F be a finitely generated non-cyclic free group, and let u and v be non-trivial elements in F which are not proper powers. Let $G = \langle F, t \mid tut^{-1} = v \rangle$ and $r \geq 2$ be a given integer. Then for particular choices of u and v there exist non-commuting elements $a, b, c \in G$ such that $a^2 b^2 c^r = 1$.*

Proof. The one-relator group $H = \langle a, b, c \mid a^2 b^2 c^r = 1 \rangle$ can also be written in terms of the presentation $\langle b, c, d \mid b^{-1} d^{-1} b = c^{-r} d \rangle$. This can be seen by letting $d = ab$ and writing the relation $a^2 b^2 c^r = 1$ as $db^{-1} db^{-1} b^2 c^r = 1$, which can then be rewritten as $b^{-1} d^{-1} b = c^{-r} d$.

Thus in the HNN extension $\langle b, c, d \mid b^{-1}d^{-1}b = c^{-r}d \rangle$ of the free group generated by $\{c, d\}$, with stable letter b and associated subgroups $\langle d \rangle$ and $\langle c^{-r}d \rangle$, the equality $a^2b^2c^r = 1$, where $a = db^{-1}$, will be satisfied, but none of a, b, c will commute.

We can clearly take the HNN extension of any finitely generated non-cyclic free group F with associated cyclic subgroups of the form $\langle x \rangle$ and $\langle y^r x \rangle$, where x and y are generators of F , and an equality of the form $a^2b^2c^r = 1$ will be satisfied without any of a, b, c commuting. \square

For brevity we refer the reader to [2] for a complete account of Λ -trees and the groups that act freely on them, called Λ -free groups.

Corollary 1. *There exist Λ -free groups in which*

$$a^2b^2c^r = 1$$

holds for non-commuting $a, b, c \in G$, and $r \geq 2$.

Proof. In [10, Theorem 3.1] it is shown that groups with a presentation of the form $\langle x, y, x_1, \dots, x_n \mid xyx^{-1}y^e = w \rangle$, where w is any word in $\{x_1, \dots, x_n\}$, act freely on $(\mathbb{Z} \times \mathbb{Z})$ -trees, unless $e = 1$ and $w = 1$. The groups arising in the proof of Theorem 1, i.e. the groups $\langle b, c, d \mid b^{-1}d^{-1}b = c^{-r}d \rangle$ where $r \geq 2$, have this form and will therefore act on a Λ -tree. They also satisfy the equation $a^2b^2c^r = 1$, where $a = db^{-1}$, with non-commuting a, b, c . \square

Before we state the next results we need to make the following observations. Let F be a free group with basis X . We remind the reader (see for example [8, Chapter I.4]) that a *Whitehead automorphism* of F is an automorphism τ of one of the following two kinds:

- (1) τ permutes the elements of $X^{\pm 1} = X \cup X^{-1}$;
- (2) for some fixed $a \in X^{\pm 1}$, τ carries each of the elements $x \in X^{\pm 1}$ into one of $x, a^{-1}x, xa$ or $a^{-1}xa$.

We now consider the special situation with $X = \{x_1, x_2, x_3\}$ and F free on X . In F we consider the two words

- (1) $w_1(x_1, x_2, x_3) = x_1^p x_2^q x_3^r$ with $p \geq 2, q \geq 3, r \geq 3$;
- (2) $w_2(x_1, x_2, x_3) = g_1 x_3 u^{\alpha_1} x_3^{-1} g_2 x_3 u^{\alpha_2} x_3^{-1} \dots g_n x_3 u^{\alpha_n} x_3^{-1}$ with $n \geq 1, 1 \neq u = u(x_1, x_2)$ and u is not conjugate to a power of x_1 or x_2 , α_i non-zero integers and g_1, \dots, g_n freely reduced words in $\langle x_1, x_2 \rangle$, with $g_i \neq 1$ for $i = 1, \dots, n$.

Remark 1. Assume that $p, q, r \geq 3$. We first note that $w_1 \neq w_2$ and w_1 is *minimal* (with respect to length) in its automorphic orbit. It can be easily seen that if τ is a Whitehead automorphism of F of type (2), if we apply τ to $w_1(x_1, x_2, x_3)$, then the length strictly increases. The minimality of w_1 also shows that it cannot be a primitive

element by [8, Proposition 4.17]. In fact, w_1 is a word of minimal rank, also called a *regular* word, that is, there is no Nielsen transformation from $\{x_1, x_2, x_3\}$ to a system d, f, g with $x_1^p x_2^q x_3^r \in \{d, f\}$ (see [6]).

Remark 2. We now consider $w_2(x_1, x_2, x_3)$. If w_2 is minimal, then there is no Whitehead automorphism τ such that the length strictly decreases when applying τ . Hence, if w_2 is minimal, there is no automorphism taking w_1 to w_2 by [8, Proposition 4.17], as the only automorphism taking w_1 to w_2 would be a permutation, and the form of the two words does not allow for a permutation to send w_1 to w_2 . If w_2 is not minimal, then each Whitehead automorphism which decreases the length of w_2 will take w_2 to a word of the same form. To see this, notice that u contains both x_1 and x_2 . If, for instance, $g_1 = g'_1 x_2$, then an automorphism which replaces x_2 by $x_2 x_3 = x'_2$ gives $x'_2 x_3^{-1}$ at all other places where x_2 occurred, especially inside u .

Remark 3. If $p = 2$ and $q, r \geq 3$, then w_1 is still minimal, and when we apply a Whitehead automorphism τ to $w_1(x_1, x_2, x_3)$ the length strictly increases, except when τ is of the form $x_1 \rightarrow x_1 x_2^{-1}, x_2 \rightarrow x_2, x_3 \rightarrow x_3$, in which case the length stays the same. If w_2 is minimal, the only automorphisms that could take w_1 to w_2 are of the form τ composed with permutations, and one can see that such automorphisms cannot take w_1 to a word of the form w_2 . As in Remark 2, if w_2 is not minimal, then each Whitehead automorphism which decreases the length of w_2 will take w_2 to a word of the same form.

From the minimality of w_1 and the facts about w_2 in the above paragraphs we get the following.

Lemma 1. *Let F be free with basis $\{x_1, x_2, x_3\}$ and $w = w_1(x_1, x_2, x_3) = x_1^p x_2^q x_3^r$ with $p \geq 2, q \geq 3, r \geq 3$. Then there is no automorphism α of F with $\alpha(x_i) = y_i, i = 1, 2, 3$, such that $\alpha^{-1}(w)$ is, written in y_1, y_2, y_3 , of the form*

$$g_1 y_3 u^{\alpha_1} y_3^{-1} g_2 y_3 u^{\alpha_2} y_3^{-1} \dots g_n y_3 u^{\alpha_n} y_3^{-1}$$

with $n \geq 1, 1 \neq u = u(y_1, y_2)$ and u not conjugate to a power of y_1 or y_2 , all α_i non-zero integers, and g_1, \dots, g_n freely reduced words in $\langle y_1, y_2 \rangle$, with $g_i \neq 1$ for $i = 1, \dots, n$.

Lemma 1 states that words of the form w_1 and w_2 cannot be in the same automorphic orbit.

Theorem 2. *Let F be a finitely generated free group, u and v non-trivial elements in F that are not proper powers, and $G = \langle F, t \mid tut^{-1} = v \rangle$. Let $a, b, c \in G$ satisfy $a^p b^q c^r = 1$ with $p \geq 2, q \geq 3, r \geq 3$.*

- (i) *If u is not conjugate to v^{-1} , then a, b, c must commute.*
- (ii) *If u is conjugate to v^{-1} , then a, b, c either commute or generate the Klein bottle group $\langle x, y \mid xyx^{-1}y = 1 \rangle$.*

Proof. Let $H = \langle a, b, c \rangle$. We will consider three cases:

- (1) u and $v^{\pm 1}$ are not conjugate;
- (2) u and v are conjugate;
- (3) u and v^{-1} are conjugate.

In case (1) assume that H is not abelian. Then H cannot be free of rank 2 or 3 because the word $a^p b^q c^r$ is regular by Remark 1. Thus, by [4, Theorem 2.2], H must be a one-relator group, that is, in our case, $H = \langle a, b, c \mid a^p b^q c^r = 1 \rangle$. Furthermore, by the proof of [4, Theorem 2.2], there is a Nielsen transformation from $\{a, b, c\}$ to a system $\{x_1, x_2, x_3\}$ for which we may assume, without loss of generality, that $x_1, x_2 \in F$, $\langle x_1, x_2 \rangle$ non-cyclic, $x_3 = t$ and H has a presentation of the form

$$H = \langle x_1, x_2, x_3 \mid w(x_1, x_2, x_3) = 1 \rangle,$$

with $w(x_1, x_2, x_3) = g_1 x_3 h^{\alpha_1} x_3^{-1} g_2 x_3 h^{\alpha_2} \dots g_n x_3 h^{\alpha_n} x_3^{-1}$, $n \geq 1$, $h = u^\alpha \in \langle x_1, x_2 \rangle$, $\alpha \neq 0$, $\alpha_i \neq 0$ and $g_i \in \langle x_1, x_2 \rangle$ non-trivial and freely reduced for $i = 1, \dots, n$. But this contradicts Lemma 1 if h is not conjugate to a power of x_1 or x_2 . Therefore H is abelian if h is not conjugate to a power of x_1 or x_2 .

Now suppose that h is conjugate to a power of x_1 or x_2 . Without loss of generality we may assume that $h = x_1^\gamma$. Since $h = u^\alpha$ and u is not a proper power in F we get that $x_1 = u^\delta$ in F . With respect to the equation $a^p b^q c^r = 1$ and because $H = \langle a, b, c \rangle$ we may replace x_1 by u . Hence, let $x_1 = u$. We may also assume that x_2 is not a proper power in F . Now let both x_1 and x_2 be not a proper power in F . Using this and the cancellation arguments in [4, Theorems 1 and 2], we see that v is in $\langle x_1, x_2 \rangle$ because H is not free of rank 2 or 3. Hence, H has a presentation of the form $K = \langle x_1, x_2, t \mid tx_1 t^{-1} = v \rangle$ with $v = v(x_1, x_2)$ freely reduced in x_1 and x_2 , and $t = x_3$. But in the free group on a, b, c there is no automorphism φ with $\varphi(a) = x_1$, $\varphi(b) = x_2$ and $\varphi(c) = t$ such that $\varphi^{-1}(a^p b^q c^r)$ with $2 \leq p, 3 \leq q, 3 \leq r$ is, written in x_1, x_2 and t , of the form $tx_1 t^{-1} v^{-1}$. This gives a contradiction. Hence, H is abelian in this case as well.

In case (2), we can assume without loss of generality that $u = v$. Then G is fully residually free, which implies that H is fully residually free. Thus H has the same universal theory as that of free groups. Therefore $a^p b^q c^r = 1$ with $p, q, r \geq 2$ implies that a, b and c commute.

In addition we remark that, for the case $u = v$, by the classification given in [7, Theorem 5], any non-abelian, non-free rank 3 subgroup K of G is a free rank one extension of centralizers of a free group of rank 2, that is, in our case: $K = \langle x_1, x_2, x_3 \mid x_3 h x_3^{-1} = h \rangle$ with $h \in \langle x_1, x_2 \rangle$; additionally, either h is regular and not a proper power in G or h is not regular, in which case K is isomorphic to $\langle x, y, \mid xy = yx \rangle \star \langle z \mid \rangle$ (the free product of a free abelian group of rank 2 and the integers).

In case (3), we can assume without loss of generality that $u = v^{-1}$, that is, $G = \langle F, t \mid tut^{-1} = u^{-1} \rangle$. Since $t^2 u t^{-2} = u$, one can easily extend the arguments in [7, Theorem 5] (which only rely on the Nielsen cancellation method, and no residual

properties, in a group with relation $u = v$, in order to obtain a classification of rank 3 subgroups), regarding non-abelian, non-free rank 3 subgroups of fully residually free groups to the case $u = v^{-1}$ and obtain that any non-abelian, non-free rank 3 subgroup K of G has a presentation $K = \langle x_1, x_2, x_3 \mid x_3 h x_3^{-1} = h^\varepsilon \rangle$, with $\varepsilon = \pm 1$ and $h \in \langle x_1, x_2 \rangle$; additionally, either h is regular and not a proper power in G or h is not regular, in which case K is isomorphic to $\langle x, y \mid x y x^{-1} = y^\varepsilon \rangle \star \langle z \mid \rangle$, where $\varepsilon = 1$ or $\varepsilon = -1$.

We assume now that $H = \langle a, b, c \rangle$ is not free, not abelian and of rank 3. Then H is isomorphic to a subgroup K of G as described above, that is, $H \cong K = \langle x_1, x_2, x_3 \mid x_3 h x_3^{-1} = h^\varepsilon \rangle$. If $\varepsilon = 1$, then H is fully residually free and so $a^p b^q c^r = 1$ implies that a, b and c commute.

Now let $\varepsilon = -1$. In case h is regular, and not a proper power, then, if H is non-free and not abelian, by the above arguments H has to be a one-relator group, that is, in our case, $H = \langle a, b, c \mid a^p b^q c^r = 1 \rangle$ since $a^p b^q c^r$ is regular. By the same arguments as in case (1), there must be a Nielsen transformation from $\{a, b, c\}$ to a system $\{x_1, x_2, x_3\}$ for which, without loss of generality, H has a presentation of the form $\langle x_1, x_2, x_3 \mid x_3 h x_3^{-1} h = 1 \rangle$. But $x_3 h x_3^{-1} h$ is a word of the form w_2 , and so by Lemma 1 this cannot happen.

If h is not regular, then $H \cong K = \langle x, y \mid x y x^{-1} = y^{-1} \rangle \star \langle z \mid \rangle$. But since a, b, c satisfy $a^p b^q c^r = 1$ and $a^p b^q c^r$ is regular, [6, Theorem 5.2] implies that H must in fact be a rank 2 subgroup, which is not the case.

Finally, let $H = \langle a, b, c \rangle$ be a non-abelian, non-free rank 2 subgroup. Then by [3, Theorem 1] H must be the Klein bottle group

$$V = \langle x, y \mid x y x^{-1} = y^{-1} \rangle \cong \langle u, v \mid u^2 = v^2 \rangle.$$

In V the elements u and v do not commute. Thus by taking, for instance, $a = u$, $b = u^{-1}$ and $c = v^{-1}$, since $u^{12} u^{-8} = v^4$, one gets that $a^{12} b^8 c^4 = 1$. \square

The groups in Theorem 2 for which the translation lengths of u and v are equal are in fact Λ -free groups by Bass' work (see [10, Theorem 2.4.1]). We may extend Theorem 2 to the following result, after reminding the reader that a group G is called n -free for a positive integer n if every subgroup of G generated by n elements is free.

Theorem 3. *Let L be a non-cyclic, 2-free, fully residually free group, u and v non-trivial elements in L that are not proper powers and u is not conjugate to v^{-1} . Let $G = \langle L, t \mid t u t^{-1} = v \rangle$. If $a, b, c \in G$ satisfy $a^p b^q c^r = 1$ for $p \geq 2, q \geq 3, r \geq 3$ then a, b, c must commute.*

Proof. We first remark that L is also 3-free by [7].

If u is conjugate to v , then we may assume that $u = v$. Then G is fully residually free and hence Theorem 3 holds.

From now on we assume that u is not conjugate to v . In the proof of Theorem 2 we used the classification of the rank 3 subgroups of G for the case that L is a non-

abelian free group (see [4]). In [4], in addition to the standard Nielsen cancellation method in HNN groups, one only needs three properties of L and G respectively:

- (1) the subgroups $\langle u \rangle$ and $\langle v \rangle$ are malnormal in L ;
- (2) L is 3-free;
- (3) each two-generator subgroup of G is free.

Now let L , as in the statement of Theorem 3, be a non-cyclic, 2-free, fully residually free group. We have to show that the properties (1), (2) and (3) also hold in this more general situation.

(1) holds because L is 2-free. Let $x \in L$ be such that $xu^\alpha x^{-1} = u^\beta$ for some integers $\alpha, \beta \neq 0$. Since L is 2-free, the subgroup $\langle x, u \rangle$ of L is cyclic. Hence $x \in \langle u \rangle$.

(2) holds by the above remark that L is 3-free.

We now show that (3) also holds. In [3], in the special case that L is a non-abelian free group we have used, besides the Nielsen cancellation method in HNN groups and property (1), only the fact that L is 2-free. But this we assume anyway for L . Hence (3) also holds for the more general situation. We may now apply analogous arguments to the ones in the proof of Theorem 2. \square

Corollary 2. *Let $S = \langle a_1, b_1, \dots, a_n, b_n \mid [a_1, b_1] \dots [a_n, b_n] = 1 \rangle$, $n \geq 2$, be an orientable surface group of genus ≥ 2 or $S = \langle a_1, \dots, a_n \mid a_1^2 \dots a_n^2 = 1 \rangle$ a non-orientable surface group of genus ≥ 4 . Let u and v be non-trivial elements in S that are not proper powers and u is not conjugate to v^{-1} , and let $G = \langle S, t \mid tut^{-1} = v \rangle$. Then if for $a, b, c \in G$ and $p \geq 2$, $q \geq 3$, $r \geq 3$ the equality $a^p b^q c^r = 1$ holds, the elements a, b, c must commute.*

Proof. In both cases S is a non-cyclic, 2-free, fully residually free group. Hence Corollary 2 holds by Theorem 3. \square

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