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INVARIANTS OF SPIN THREE-MANIFOLDS FROM CHERN-SIMONS THEORY AND FINITE-DIMENSIONAL HOPF ALGEBRAS

STEPHEN F. SAWIN

ABSTRACT. A version of Kirby calculus for spin and framed three-manifolds is given and is used to construct invariants of spin and framed three-manifolds in two situations. The first is ribbon $*$ -categories which possess odd degenerate objects. This case includes the quantum group situations corresponding to the half-integer level Chern-Simons theories conjectured to give spin TQFTs by Dijkgraaf and Witten [10]. In particular, the spin invariants constructed by Kirby and Melvin [21] are shown to be identical to the invariants associated to $SO(3)$. Second, an invariant of spin manifolds analogous to the Hennings invariant is constructed beginning with an arbitrary factorizable, unimodular quasitriangular Hopf algebra. In particular a framed manifold invariant is associated to every finite-dimensional Hopf algebra via its quantum double, and is conjectured to be identical to Kuperberg's noninvolutory invariant of framed manifolds associated to that Hopf algebra.

INTRODUCTION

This article is motivated by, and addresses, two separate questions. The first is Dijkgraaf and Witten's remarkable observation in [10] based on the path-integral formulation. They argue that for certain nonsimply-connected Lie groups G the level k in the definition of the Chern-Simons field theory, which ordinarily must be an integer in order to get a well-defined topological quantum field theory, can be a half-integer and still be expected to yield a sensible theory in the spin category. One would like to be able to reproduce this observation rigorously in the algebraic/combinatorial quantum group formulation, as part of a general effort to relate these two settings.

The second question has to do with the mysterious 'nonsemisimple' topological invariants coming from quantum groups discovered by Kuperberg [25] (generalizing the semisimple version studied by many authors including Kuperberg [24], Barrett and Westbury [3, 4], and Chung, Fukuma, and Shapere [9]) and Hennings [13] (also considered by Lyubashenko [27, 26], Kerler [17, 18], and Kauffman and Radford

[15, 16]), which appear to be closely related to, but distinctly different from, the Chern-Simons invariants. The Kuperberg invariant, which assigns an invariant of framed three-manifolds to each finite-dimensional Hopf algebra H , is widely conjectured to be identical to the Hennings invariant associated to the quantum double of H . Unfortunately, for a typical Hopf algebra the quantum double is quasitriangular, but not necessarily ribbon, while the Hennings invariant requires a ribbon Hopf algebra (with some additional nondegeneracy conditions). This additional structure is reflected in the fact that the Hennings invariant depends only on a 2-framing (which can be normalized away with some loss in terms of cutting and pasting relations), rather than the more involved framing. Thus one would like to extend the Hennings invariant to associate framing-dependent invariants to quasitriangular Hopf algebras. This is especially important because the framing structure represents much of the complexity that makes the Kuperberg invariant difficult to work with.

This article solves both of these questions in a common framework. Relying on the fact that, crudely speaking, a framing is a spin structure plus a 2-framing, we identify in both the Chern-Simons and non-semisimple case the weakening of algebraic structure from ribbon to quasitriangular with a weakening of the topological invariance from dependent on a 2-framing to dependent on a framing. This identification can be traced ultimately to the link invariant level, where in the quasitriangular case the natural quantities that arise are invariants only of links with even framing or self-linking number.

As interesting as the two questions are separately, the connection between them revealed by this common framework deserves attention also. The hints of the geometry of Chern-Simons theory which pervades the Kuperberg and Hennings invariants seem to demand a physical explanation, and it is to be hoped that a link between Kuperberg and Hennings' algebra on the one hand and Dijkgraaf and Witten's geometry on the other offers a useful step towards such an explanation.

Section 1 gives a combinatorial description of framed three-manifolds (i.e., equipped with a spin structure and an even 2-framing) in terms of surgery on links all of whose components have even framings. Section 2 gives a general framework analogous to Reshetikhin and Turaev's modular Hopf algebras [33] for constructing invariants of spin manifolds and identifies the invariants of this sort arising from quantum groups with the Chern-Simons theories meeting Dijkgraaf and Witten's conditions for spin theories. Also, the $SO(3)$ theory is identified with Kirby and Melvin's [21] spin invariants constructed from quantum \mathfrak{su}_2 , and a formulation generalizing theirs is given in general. Finally Section 3

defines a Hennings-type invariant of spin (or framed) three-manifolds starting from a factorizable quasitriangular Hopf algebra (these conditions include the quantum double as a special case). Below we offer a more detailed introduction to each of the three sections.

Even links, spin three-manifolds and surgery. Our approach to topological invariants is the now familiar one of what is sometimes called quantum topology: Describe the topological entity by some combinatorial data modulo relations generated by a few simple ‘moves,’ translate the data to algebraic objects, and observe that, magically, the moves are algebraic relations which the objects at hand satisfy. Of course the magic reflects a hidden and poorly understood geometric underpinning, which we will discuss in the next section.

The first instance of this approach is even links: links with an even framing on each component. Here the data is a link projection with each component having winding number one, and the moves are Kauffman’s regular isotopy, weaker than the usual Reidemeister or framed Reidemeister moves. In the translation of these moves to the language of Hopf algebras we will see that the ribbon conditions are no longer necessary. The use of winding number one rather than zero corresponds on the algebraic side to evaluating with quantum characters rather than tracial functions, which is fairly natural from the point of view of Hopf algebras themselves but does not fit neatly in the language of rigid braided categories.

Surgery on framed links produces four-manifolds with three-manifold boundary. With the appropriate moves (2-framed Kirby moves) surgery gives a combinatorial description of three-manifolds equipped with the 2-framing needed to regularize Chern-Simons theory (see Atiyah [2], and [35]). Surgery on even links produce spin four-manifolds with spin three-manifold boundary. With the appropriate moves (spin Kirby moves) such an even surgery give a combinatorial description of spin three-manifolds equipped with a compatible 2-framing. Specifically, the 2-framing has to equal Rohlin’s μ invariant modulo 16. Since framings of three-manifolds are in one-to-one correspondence with spin structures together with 2-framings, we get a combinatorial description of framed manifolds, but the compatibility puts a constraint on the possible framings. The exact significance of this restriction as it relates to the framings of Kuperberg’s invariant is not clear, although since a fixed shift in the 2-framing multiplies each invariant by a fixed quantity, the distinction is fairly minor. This section also offers a Fenn-Rourke version of the spin Kirby moves which replaces the general handle-slides with a more restrictive ‘semilocal’ move.

Reshetikhin-Turaev type spin invariants. Ribbon categories (see Reshetikhin and Turaev [33], Turaev [39], or Kassel [14]) are the appropriate place to look for framed link invariants, and if they meet some additional conditions (summarized in the definition of a modular category in [39]) they give invariants of 2-framed three-manifolds through the surgery presentation. One might think that modularity is a very restrictive constraint on a ribbon category, but in fact it is not. Müger [28] and Bruguières [7] give a kind of quotient of a ribbon $*$ -category (Bruguières works with a more general but perhaps less natural substitute for the $*$ -structure) that roughly speaking deletes the part of the category which provides no link information. Sometimes this quotient results in a modular category. We say sometimes because the objects which stand in the way of modularity (called degenerate) come in two flavors, odd and even, and only the even can be eliminated. Thus the existence of odd degenerate objects is the only obstruction to constructing a modular category from a ribbon category.

The quotient fails to go through for odd degenerate objects because they actually carry some information about the link. In fact they contribute a factor of -1 raised to the self-linking number for every component they label. Of course if we restrict our attention to even links, there is no information at all, and the quotient can go through. Imitation of the usual Reshetikhin-Turaev construction in this case yields an invariant of spin and compatibly-framed three-manifolds (presumably it yields an appropriately modified version of the axioms of topological quantum field theory, but we defer that important question and focus only on the invariants in this article). In [37] the author analyzed the quotient construction applied to subsets of the Weyl alcove. This analysis allows us to identify the levels at which we get a spin Chern-Simons theory associated to a given simple Lie group.

On the geometric side, we generalize Dijkgraaf and Witten's observation, relying on the integration of the generating class of $H^4(BG, \mathbb{Z})$ on spin four-manifolds, to a classification of when the physical interpretation leads us to expect spin theories. In fact we get complete agreement with the algebraic answer (there is actually a small subtlety: as in [37], the Dijkgraaf-Witten theories often factor as a product of invariants, but the set of all these factors is a complete list of theories arising from quantum groups).

Finally, we imitate Kirby and Melvin's construction of spin invariants from quantum \mathfrak{su}_2 at certain levels to arbitrary quantum groups. In fact we find that it works in just the situations in which there are Dijkgraaf-Witten theories, and prove that the Kirby-Melvin spin invariants in fact coincide with the Dijkgraaf-Witten theories. Kirby

and Melvin did their computation in quantum \mathfrak{su}_2 , and thus associated their invariant with $SU(2)$, while we do the computation entirely in the set of representations associated to $SO(3)$, and thus find the invariant more naturally associated to $SO(3)$ as expected from the geometry and physics.

Hennings-type invariants of spin manifolds. In the combinatorial version of Chern-Simons and the other traditional quantum invariants, one does not work with the quantum group itself, which is not semisimple, nor with the whole of the representation theory. Instead one works with a piece of the representation theory modified so as to resemble the representation theory of a semisimple ribbon Hopf algebra. The theory outside the Weyl alcove, which roughly corresponds to the nonsemisimple part of the quantum group, is simply thrown away.

The Hennings invariant, by contrast, relies heavily on the nonsemisimple part of the quantum group. More precisely, the link invariants naturally associated to the three-manifold invariant are labeled by quantum characters, which include the quantum traces of irreducible representations labeling links in Reshetikhin and Turaev approach, but also include functions associated to nonirreducible representations. What's more the analogue of the surgery label (called in the sequel ω) which Reshetikhin and Turaev use to construct the three-manifold invariant is the left integral. This integral turns out to be an element of the socle of the algebra of quantum characters (roughly, a maximally nilpotent element). The topological effect of this nonsemisimplicity is that the invariant is zero except for rational homology three-spheres and satisfies only a subset of the cut-and-paste axioms of TQFT which its semisimple cousins satisfy.

We find for a quasitriangular Hopf algebra which is not necessarily ribbon that quantum characters label invariants of even links (In the presence of a ribbon element, quantum characters and cocommutative functionals can be used almost interchangeably, but in our more delicate situation we find that quantum characters play the more fundamental role). If the algebra is unimodular, the left integral in the dual (which is also the right integral) is a quantum character, and in the presence of a nondegeneracy condition (factorizability suffices) it has the appropriate properties to give an invariant of compatibly framed three-manifolds. The quantum double of a finite-dimensional Hopf algebra meets both conditions (factorizable and unimodular), though it is often not ribbon, and we conjecture that the invariant of framed manifolds which we associate to the quantum double of H is the framed invariant Kuperberg associates by very different means to H . We also

conjecture that the Dijkgraaf and Witten spin invariants of the previous section arise by a construction analogous to that of Reshetikhin and Turaev from a quasitriangular but not ribbon Hopf algebra.

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1. EVEN LINKS, SPIN THREE-MANIFOLDS AND SURGERY

1.1. Even Links. A framed link is an oriented link in oriented S^3 together with a nonzero section of the normal bundle, considered up to ambient isotopy. The framing or self-linking number of a component is the linking number between that component and a copy of it pushed off slightly in the direction of the framing. An *even* link is a framed link all of whose components have even self-linking number. A projection of a framed link (in particular of an even framed link) is a projection of any representative of the isotopy class onto the oriented plane (the representative is first viewed as sitting inside \mathbb{R}^3) such that the projection of the link is a smooth immersion of the union of circles with no self-intersections other than transverse double points, and the framing is never orthogonal to the projection. Of course such a projection, together with identification of each crossing as over or under, uniquely determines the isomorphism class of the framed link. We will sometimes be interested in such projections which also come equipped with a *height function*, i.e., a smooth map from \mathbb{R}^2 to \mathbb{R} without critical points such that when the map is restricted to the projection its critical points are nondegenerate and do not fall on the crossings.

An isotopy of a framed link and a projection (and of a height function) is collectively called a *planar isotopy* (respectively *simple isotopy*) if at each point of the isotopy the link projection (and height function) satisfy the conditions of the previous paragraph.

The winding number of a component of a projection is the total number of complete counterclockwise rotations the tangent vector to that component of the projection undergoes in a complete circuit around the component in the direction of its orientation. It is also half the total signed number of critical points of the height function, with those turning counterclockwise counting as $+1$ and those turning clockwise counting as -1 .

Note that the winding number of a component of a projection and the framing of that projection are always of opposite parity (i.e. one is odd, the other is even). To see this notice each changes parity only when the other does under the Reidemeister moves (see Burde and

Zieschang [8]), so if the claim is true for one projection of a link it is true for all projections of that link. It is certainly true for a component which is a framed unknot unlinked with the other components (using the simple projection in which it participates in no crossings with other components and crosses with itself n times where n is the absolute value of the framing) and it remains true when a single crossing in a projection is switched from over to under or under to over. Since it is well known (see Adams [1], for example) that one can get from any projection to a projection of unlinked unknots by a sequence of such crossing changes, the claim is true in all projections.

Proposition 1. *Every link admits a projection in which each component with even framing has winding number one and each component with odd framing has winding number zero. Two such projections are of the same framed link if and only if they can be connected by planar isotopy together with the regular isotopy moves shown in Figure 1 (understood to apply with any orientations on the pictured strands). Two such projections equipped with height functions can be connected by simple isotopy together with the regular isotopy moves (with the vertical indicating the height function) and the height function moves in Figure 2.*



FIGURE 1. The moves of regular isotopy

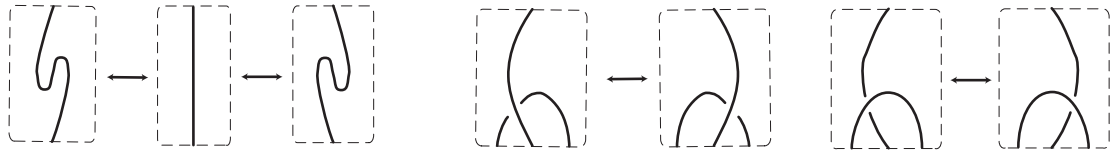


FIGURE 2. Moves which change the height function

Proof. Recall every framed link admits a projection (every link admits a projection according to [8], and by adding a suitable number of full twists to the projection one can make the projection have any given framing), so choose one such. Notice the moves in Figure 3 do not

change the framed link, but change the winding number of the indicated component by ± 2 . A sequence of such moves gives a projection in which every component has winding number one or zero, and the observation preceding the proposition indicates these components must have respectively even or odd framings.

For the second claim, Trace [38] has shown that two link projections are regular isotopic (i.e., connected by a sequence of regular isotopy moves and planar isotopy), if and only if their links are isotopic and each component has the same writhe and winding number. The writhe of a component of a projection is the signed sum of the crossings of the component with itself, which is of course the framing if it is a projection of a framed link.

That every framed link admits a height function is clear, by choosing any real function with no critical points and perturbing slightly as necessary. That two such are connected by a sequence of the height function moves appears in Kassel [14, Lemma X.3.5]. \square

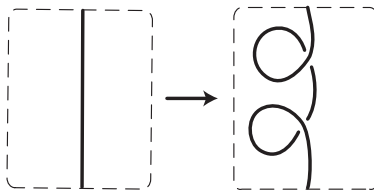


FIGURE 3. Changing the winding number without changing the framing

Remark 1. The appropriate moves for framed links without the restriction on winding number are the regular isotopy moves of Figure 1 together with a move that replaces a positive full twist with winding number $+1$ with a positive full twist with winding number -1 . These are called the framed Reidemeister moves [14, 36]. The regular isotopy moves are strictly weaker.

In Section 3 we will assign numbers to link projections with each component having winding number one, such that the numbers are unchanged by simple isotopy and the moves of regular isotopy, and thus are invariants of the even links the projections represent.

1.2. Framed and Spin Three-Manifolds. Recall that the ordinary Chern-Simons invariant depends not simply on a three-manifold but on a 2-framed three-manifold. A 2-framing [2, 35] of a three-manifold M can either be defined as a trivialization up to isotopy of a $\text{Spin}(6)$

bundle associated with $TM \oplus TM$, or as a choice of cobordism class of four-manifolds which M bounds [19]. We will use the second definition, and recall that the cobordism class of a four-manifold is determined by its boundary and signature, so we may think of a 2-framing on M as an integer representing the signature of the four-manifold [19].

The choice of a spin structure (recall every compact oriented three-manifold admits a spin structure, defined as a lifting up to isotopy of the $SO(3)$ bundle associated to TM to an $SU(2)$ bundle) and a 2-framing for M is equivalent to the choice of a framing for M , i.e. a trivialization of the tangent bundle up to isotopy [35]. That is, if two framings induce the same spin structure and 2-framing they are homotopic, and every combination of spin structure and 2-framing is induced by some framing.

According to Kirby [22, Chapters II, IV, and VII] every spin three-manifold spin bounds a spin four-manifold, in fact a 2-handlebody (i.e., a four-manifold formed by attaching a collection of 2-handles to a 0-handle). A 2-handlebody is spin if and only if its intersection form on second cohomology is even, and in this case possesses a unique spin structure. Thus we may specify a spin structure on a three-manifold by specifying a 2-handlebody with even form which the three-manifold bounds. Now a 2-framing on a three-manifold can also be determined by specifying a four-manifold which it bounds, in fact it is determined by the signature of the intersection form (again, it may as well be a 2-handlebody). It is natural to represent both pieces of information by a single four-manifold, but this is only possible for certain framings. Recall [22, Chapter XI] that if M is a spin three-manifold then Rohlin's invariant $\mu(M)$ is an even integer modulo 16 such that every four-manifold which spin bounds M has signature equal to $\mu(M)$ modulo 16. Motivated by this we define

Definition 1. *A framing on a compact, connected, oriented three-manifold M is called compatible if the induced 2-framing as an integer is equal to $\mu(M) \pmod{16}$.*

Thus a compatible framing on M can be represented by a 2-handlebody.

1.3. Spin Manifolds and Surgery. Recall from Kirby [22, Chapter I] that if W is a 2-handlebody, W can be presented by an unoriented framed link in S^3 . Here S^3 bounds the 0-handle B^4 , and the link with each component thickened to a solid torus with distinguished longitude determines how to attach a 2-handle along each component. The matrix of the intersection form is given by the linking matrix, which is even (i.e. has even entries along the diagonal) if and only if the

2-handlebody is spin. The boundary of a four-manifold described by a link is the three-manifold obtained by surgery on that link. That is, it is the result of removing a tubular neighborhood of each component and gluing it back in by sending the meridian to the longitude and the longitude to minus the meridian. Thus surgery on an even link can be viewed as resulting in not simply a three-manifold, but a three-manifold together with a compatible framing.

The following theorem and proof are direct translations of Kirby's well-known surgery theorem [20] to the spin case.

Theorem 1.

(a) *Every three-manifold with a compatible framing can be presented by surgery on an even link. Two such presentations determine the same framed three-manifold if and only if they can be connected by a sequence of the spin Kirby moves pictured in Figure 4. Here move I is the usual handleslide or band connect sum of Kirby's original theorem.*

(b) *Every spin three-manifold can be presented by surgery on an even link, and two such determine the same spin three-manifold if and only if they can be connected by a sequence of spin Kirby moves I and II as above together with distant union with the link representing the Kummer surface, pictured for example in [22, page 9].*

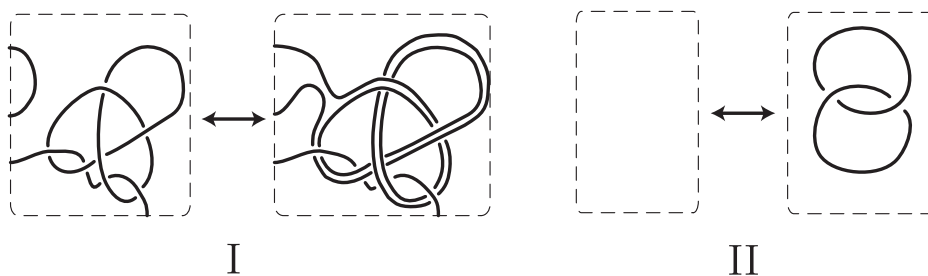


FIGURE 4. Spin Kirby moves

Proof.

(a) The first sentence is clear, as is the fact the the spin Kirby moves do not change the framed three-manifolds. So suppose M^4 and N^4 are two spin 2-handlebodies with the same spin boundary and signature presented by links L_M and L_N . We will show that L_M and L_N can be connected by spin Kirby moves.

Since they each have the same signature, gluing M to $-N$ along $\partial M \times I$ we get a closed spin manifold with signature zero. Hence by [22, Thm. VII.3] there is a spin five-manifold W which this closed manifold bounds. Choose a Morse function $f : W \rightarrow [1, 2]$ such that

$$\begin{aligned} f^{-1}[1] &= M \\ f^{-1}[2] &= N \\ f(\sigma, t) &= t \text{ for } (\sigma, t) \in \partial M \times I. \end{aligned}$$

Each $f^{-1}[t]$ for t not a critical point is a spin four-manifold. If t is an index 1 critical point then $f^{-1}[t + \epsilon]$ is $f^{-1}[t - \epsilon]$ connect summed with $S^1 \times S^3$, with the same spin structure outside the region of the connect sum. In between these two manifolds we can change W by replacing the 1-handle with a 2-handle attached along a contractible loop in the boundary. The spin structure on the boundary $f^{-1}[t \pm \epsilon]$ induces a spin structure on the boundary S^4 of the 2-handle B^5 , which extends to the interior. In this fashion W can be replaced by a new spin five-manifold in which there are no 1-handles but the Morse manifolds (i.e. f^{-1} of noncritical values) are the same. Similarly we can replace all the 4-handles with 3-handles.

Arrange f so that all 2-handles have Morse values less than those of all 3-handles. Because the Morse manifolds are simply-connected, each 2-handle as it is attached connect sums $S^2 \times S^2$ to the Morse manifold, which corresponds to spin Kirby move II applied to L_M . Likewise by flipping f we see each 3-handle corresponds to move II applied to L_N . Thus a sequence of Moves II applied to L_M and L_N gives links representing the same spin four-manifold with boundary ∂M . Kirby's argument that these can be connected by a sequence of moves I goes through unchanged.

(b) If M and N have the same spin three-manifold boundary, then their signature differs by a multiple of 16. Thus the union of one of their links with sufficiently many copies of the Kummer surface link results in two links which present the same framed manifold, and part (a) of the theorem applies. \square

Remark 2. The link invariants we construct will be multiplicative in the sense that the invariant of a distant union of links (i.e. the union of two links embedded simultaneously in S^3 so that they are separated by an S^2) is the product of their individual invariants. If a multiplicative link invariant $I(L)$ is invariant under the spin Kirby moves,

it is an invariant of compatibly framed three-manifolds. Furthermore $I(L)I(K)^{-\sigma/16}$, where K is an even link representing the Kummer surface and σ is the signature of the linking matrix of L , is invariant under the additional move of part (b) and thus gives an invariant of spin manifolds. Since the spin and framed versions of the invariant differ only by this simple normalization, we will move freely between the two versions. The two versions are exactly analogous to the ordinary and 2-framed version of the three-manifold invariant of Reshetikhin and Turaev [33]. As in that case, we expect the framed version to be more natural at the level of TQFTs.

Just as Fenn and Rourke's [12] 'semilocal' simplification of Kirby's surgery theorem gives an alternate set of moves which are sometimes more convenient for addressing three-manifold invariance, so we will find it helpful to have the following version of the spin Kirby moves at hand.

Proposition 2. *Spin Kirby moves I and II of Figure 4 generate the same equivalence relation on links as moves I' and II, where I' is pictured in Figure 5, with the number of strands passing through the unknot being arbitrary.*

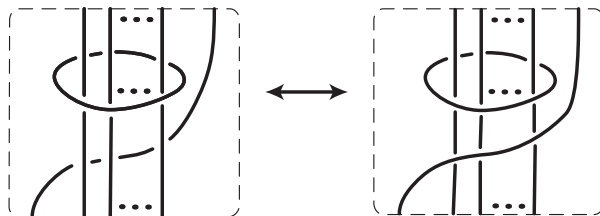


FIGURE 5. Alternate spin Kirby move I'

Proof. Of course it suffices to take an arbitrary instance of spin Kirby move I and decompose it as a sequence of moves I' and II.

Let L be an even link with a component C and let B be another component to be band connect summed with C , as illustrated in Kirby move I in Figure 4. Choose a presentation of L with the winding number of C being one as in Proposition 1, and such that the band between B and C along which the connect sum is to be applied does not overlap any component of the link. Choose n crossings of C with itself such that flipping the parities of these n crossings (i.e. over to under or vice versa) makes C a 0-framed unknot (this step relies on the winding number condition). Apply move II n times to create a Hopf link for

each of these crossings, then apply move I' twice at each crossing as in Figure 6(A). The effect of these moves is to make C a 0-framed unknot (of course the disk it bounds intersects L in many places), and in this new link, the band connected sum of B with C along the same band is an instance of move I' . After applying this band connected sum, the vicinity of each of the n crossings looks like the left-hand side of Figure 6(B), and undoing the two instances of move I' and the instance of move II at each crossing results in the right side of Figure 6(B), which is a projection of the original band connect sum of B with C . \square

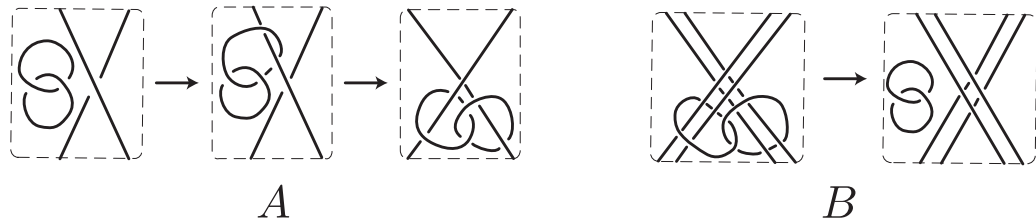


FIGURE 6. Using handle-slides with the Hopf link to switch a crossing

Remark 3. Surgery is described by an unoriented framed link while the quantities of the upcoming sections will naturally be invariants of oriented framed links. Thus our strategy will be to find an oriented link invariant which is unchanged by reversal of the orientation of any component, as well as by spin Kirby moves I' and II . Theorem 1 and Proposition 2 then say that such an invariant will actually be an invariant of the framed three-manifold presented by surgery on that link.

2. RESHETIKHIN-TURAEV TYPE SPIN INVARIANTS

2.1. Modularity and spin modularity. We review here the main results of [37] which will be relevant to the question of spin invariants.

Recall (see for example Kirillov [23], Turaev [39] and [36, 37]) that the quantum group $U_q(\mathfrak{g})$, where \mathfrak{g} is a simple Lie algebra and q is a root of unity, forms a ribbon Hopf algebra. More precisely, the set of representations of $U_q(\mathfrak{g})$ spanned by the finite collection of irreducible representations with highest weight in the Weyl alcove forms a semisimple ribbon $*$ -category with the truncated tensor product $\hat{\otimes}$. Thus we get a numerical invariant of framed graphs with edges labeled by such representations and vertices labeled by invariant elements of an appropriate tensor product of the labels of incident edges and the duals of

those labels (invariant elements are in general morphisms in the ribbon category) such that the following hold:

1. the invariant of a graph with an edge labeled by $\lambda \oplus \gamma$ is the sum of the invariants of the same graph with that edge labeled by λ and γ respectively, the labels on the adjacent vertices being projected appropriately,
2. if the label of an edge is replaced by an isomorphic object and the labels of the adjacent vertices are composed with the isomorphism in the obvious way, the invariant is unchanged. In particular, link components can be unambiguously labeled by elements of Γ , rather than objects,
3. the invariant of a graph with an edge labeled by the trivial object (the weight 0) is the same as the invariant of the graph with that edge deleted,
4. the invariant of a graph with an edge labeled by λ is the invariant of the graph with the orientation of that edge reversed and the label replaced by the dual λ^\dagger , the labels of the adjacent vertices remaining the same,
5. the invariant of a graph with a link component labeled by $\lambda \hat{\otimes} \gamma$ is the invariant of the graph with that component replaced by two parallel components (according to the framing) labeled by λ and γ respectively,
6. the invariant of the connected sum of two graphs along edges labeled by a simple object λ is the product of the invariants of the two graphs divided by $\text{qdim}(\lambda)$ and
7. for any objects $\lambda_1, \dots, \lambda_n$ there is a collection of pairs of invariant elements $f_i \in \lambda_1 \hat{\otimes} \dots \hat{\otimes} \lambda_n \hat{\otimes} \gamma_i$ and $g_i \in \lambda_1^\dagger \hat{\otimes} \dots \hat{\otimes} \lambda_n^\dagger \hat{\otimes} \gamma_i^\dagger$ for various simple objects γ_i such that if L is any graph and a ball intersects L as in Figure 7, the sum of the invariants of the graphs on the right side of the figure equals the invariant of L .

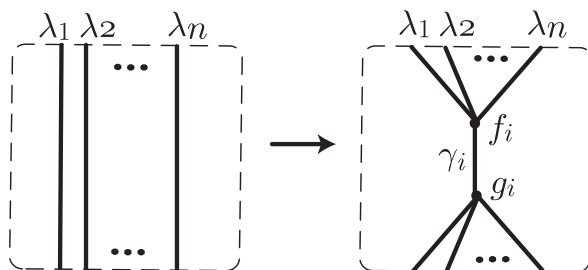


FIGURE 7. Binding edges together

The invariant of the unknot labeled by a representation λ is called the quantum dimension of λ , $\text{qdim}(\lambda)$, and the invariant of the Hopf link labeled by λ and γ is called $S_{\lambda,\gamma}$. C_λ is the modulus one complex number by which the link invariant is multiplied when a component labeled by λ is given a positive full twist.

An object λ (i.e. a representation) in the ribbon category is called *degenerate* if $R_{\lambda,\gamma} = R_{\gamma,\lambda}^{-1}$ for all objects γ , where $R_{\lambda,\gamma}$ is the morphism corresponding to a positive crossing. By necessity if λ is degenerate then $C_\lambda = \pm 1$. The set of degenerate objects with $C_\lambda = 1$ (such objects are called *even*) forms a symmetric subcategory which is isomorphic to the representation theory of some compact group, and Müger proves [28] that one can quotient by any full subcategory of this subcategory in the sense that there is a minimal semisimple ribbon $*$ -category which admits a ribbon $*$ -functor from the original category to it sending all the objects in the subcategory and only those objects to direct sums of the trivial object. In particular, if we quotient by the subcategory of *all* even objects the resulting category will have no even degenerate objects except sums of the trivial object.

Suppose \mathfrak{C} is a semisimple ribbon $*$ -category with the property that the only even degenerate simple object is trivial, and suppose that it contains a degenerate simple object μ with $C_\mu = -1$ (naturally, we call μ an *odd* degenerate object). Then $\mu \hat{\otimes} \mu$ is degenerate and even, and therefore is a sum of copies of the trivial object. Since μ is simple, this means $\mu^\dagger = \mu$ and $\mu \hat{\otimes} \mu$ is exactly the trivial object (here μ^\dagger is the dual object to μ). If ν is a degenerate object with $C_\nu = -1$, then $\mu \hat{\otimes} \nu$ is even, so it is a sum of trivial objects, so ν is a sum of objects isomorphic to μ . Thus we have the following.

Theorem 2. *Let \mathfrak{C} be a semisimple ribbon $*$ -category, and let \mathfrak{C}' be the quotient by the full symmetric subcategory of even degenerate objects as described above. Then there are two possibilities*

- (a) *If all degenerate objects in \mathfrak{C} are even, then \mathfrak{C}' contains no degenerate objects, and therefore according to Müger is modular, and can be used to construct a three-manifold invariant and TQFT following Reshetikhin and Turaev [33, 39].*
- (b) *If \mathfrak{C} contains any degenerate objects which are not even, then \mathfrak{C}' contains exactly one simple degenerate object μ , and μ satisfies $C_\mu = -1$ and $\mu \hat{\otimes} \mu$ is trivial. In this case we call \mathfrak{C}' spin modular, and will construct spin and compatibly framed three-manifold invariants from it.*

Remark 4. The $*$ -structure on the ribbon category is not necessary. Bruguières has a similar construction to Müger’s which replaces the $*$ -structure with semisimplicity together with a fairly easy to check and general condition on the degenerate objects. The $*$ -structure, which is available in all the cases of interest to us and clearly is related to the physical origin of the relevant invariants, is merely a convenience in this situation.

2.2. Framed and spin three-manifold invariants. Let \mathfrak{C} be a spin modular category, and let μ be the simple degenerate object.

Lemma 1. *For each simple object $\lambda \in \mathfrak{C}$, the object $\sigma(\lambda) \stackrel{\text{def}}{=} \mu \hat{\otimes} \lambda$ is a simple object distinct from λ , and in any link with an even framed component labeled by λ , the invariant is unchanged if that label is replaced by $\sigma(\lambda)$. In particular, if V is the vector space of formal linear combinations of simple objects in \mathfrak{C} (the link invariant extends by linearity to links labeled by elements of V) then σ extends to a \mathbb{Z}_2 action on V , the link invariant descends to an invariant of even links labeled by elements of V/σ , the map $V \rightarrow V/\sigma$ preserves the duality map \dagger (thought of as another \mathbb{Z}_2 action on V) and finally the S -matrix gives a nondegenerate pairing on V/σ .*

Proof. Since $\mu \hat{\otimes} (\mu \hat{\otimes} \lambda) = \lambda$ is simple, $\mu \hat{\otimes} \lambda$ must be simple. Of course $C_{\sigma(\lambda)} = -C_\lambda$ is different from C_λ so λ and $\sigma(\lambda)$ are distinct. For any link L , the invariant of L with an even component labeled by $\mu \hat{\otimes} \lambda$ is the invariant of L with that component doubled and labeled by μ and λ respectively. Since μ is degenerate we have $R_{\mu,\gamma} = R_{\gamma,\mu}^{-1}$, so in particular in any projection of this link any crossing involving μ can have its parity switched (over to under or vice versa) without changing the invariant. A sequence of such changes can unlink and unknot the component labeled by μ , so the invariant of the doubled link is equal to the invariant of the original link time $\text{qdim}(\mu)C_\mu^n$, where n is the framing of the component labeled by μ . Now n is even and $C_\mu = -1$, so we get only the factor of $\text{qdim}(\mu)$. But since $\mu \hat{\otimes} \mu$ is trivial, we know $\text{qdim}(\mu)^2 = 1$, and since in a $*$ -category quantum dimensions are positive, $\text{qdim}(\mu) = 1$.

Of course σ extends to a \mathbb{Z}_2 action on V by linearity, which commutes with the duality map \dagger . Again by linearity it is true that labeling a component by any $v \in V$ gives the same value to the invariant of an even link as labeling it by $\sigma(v)$, so we can as well label components of an even link by equivalence classes $\{v, \sigma(v)\} \in V/\sigma$. Thus the pairing defined by $\langle \lambda, \gamma \rangle = S_{\lambda,\gamma}$, descends to a well-defined pairing on V/σ .

Now the tensor product $\hat{\otimes}$ extends by linearity to an associative, distributive multiplication with identity on V (here we identify $\lambda \oplus \gamma$ with the vector $\lambda + \gamma$ of V), thus making V an algebra. Since

$$\sigma(\lambda) \hat{\otimes} \gamma = \sigma(\lambda \hat{\otimes} \gamma) = \lambda \hat{\otimes} \sigma(\gamma),$$

the quotient V/σ inherits the algebra structure.

For each equivalence class $[\lambda]$ of a simple object λ notice $f_{[\lambda]}([\gamma]) \stackrel{\text{def}}{=} \langle \lambda, \gamma \rangle / \text{qdim}(\lambda)$ is a nontrivial homomorphism from V/σ to \mathbb{C} . Supposing the pairing $\langle \cdot, \cdot \rangle$ is degenerate, then these $\dim(V)/2$ homomorphisms must be linearly dependent, and thus two must be equal. If $f_{[\lambda]} = f_{[\lambda']}$, then $\langle \lambda, \gamma \rangle / \text{qdim}(\lambda) = \langle \lambda', \gamma \rangle / \text{qdim}(\lambda')$ for all γ . Now since $\langle \lambda' \hat{\otimes} \lambda^\dagger, \gamma \rangle = \langle \lambda', \gamma \rangle \langle \lambda^\dagger, \gamma \rangle / \text{qdim}(\gamma)$ we have

$$\langle \lambda \hat{\otimes} \lambda^\dagger, \gamma \rangle / \text{qdim}(\lambda) = \langle \lambda' \hat{\otimes} \lambda^\dagger, \gamma \rangle / \text{qdim}(\lambda')$$

for all γ . In particular, since $f_{[\text{id}]}$ (where id is the trivial object, which is the multiplicative identity) is nontrivial there is a minimal idempotent ω such that $f_{[\text{id}]}(\omega) = 1$ but $f_{[\gamma]}(\omega) = 0$ if $f_{[\gamma]} \neq f_{[\text{id}]}$. Now

$$\langle \lambda \hat{\otimes} \lambda^\dagger, \omega \rangle = \sum_{\gamma} N_{\lambda, \lambda^\dagger}^{\gamma} \langle \gamma, \omega \rangle \geq N_{\lambda, \lambda^\dagger}^{\text{id}} = 1,$$

where $N_{\lambda, \lambda^\dagger}^{\delta}$ is the multiplicity of δ in $\lambda \hat{\otimes} \lambda^\dagger$. Thus

$$\langle \lambda' \hat{\otimes} \lambda^\dagger, \omega \rangle \geq \text{qdim}(\lambda') / \text{qdim}(\lambda) > 0$$

so there exists a γ such that $N_{\lambda', \lambda^\dagger}^{\gamma} = 1$ and $f_{[\gamma]} = f_{[\text{id}]}$.

Of course if $[\gamma] = [\text{id}]$ then $\lambda' = (\lambda^\dagger)^\dagger = \lambda$ or $\lambda' = \sigma(\lambda^\dagger)^\dagger = \sigma(\lambda)$, so since $[\lambda'] \neq [\lambda]$ we conclude that there is a $\gamma \neq \text{id}, \mu$ such that $\langle \gamma, \lambda \rangle = \text{qdim}(\gamma) \text{qdim}(\lambda)$ for all λ . Müger proves that this property implies γ is degenerate, so we reach a contradiction and conclude the pairing was nondegenerate. \square

Lemma 2. *Let $\omega = \sum_{\gamma} \text{qdim}(\gamma) \gamma$.*

- (a) *For all $v \in V$, $v \hat{\otimes} \omega = \text{qdim}(v) \omega$*
- (b) *$\langle \lambda, \omega \rangle / \text{qdim}(\lambda)$ is $\text{qdim}(\omega)$ if $\lambda = \text{id}$ or $\lambda = \mu$, and 0 otherwise.*
- (c) *$\langle \omega, \omega \rangle = 2 \text{qdim}(\omega) \neq 0$.*

Proof.

(a) It suffices to prove this for $v = \lambda$ with $\lambda \in \Lambda$.

$$\begin{aligned} \lambda \hat{\otimes} \omega &= \sum_{\gamma} \text{qdim}(\gamma) \lambda \hat{\otimes} \gamma = \sum_{\gamma, \eta} \text{qdim}(\gamma) N_{\lambda, \gamma}^{\eta} \eta \\ &= \sum_{\gamma, \eta} N_{\lambda, \eta^{\dagger}}^{\gamma^{\dagger}} \text{qdim}(\gamma) \eta = \sum_{\gamma, \eta} N_{\lambda, \eta^{\dagger}}^{\gamma^{\dagger}} \text{qdim}(\gamma^{\dagger}) \eta \\ &= \sum_{\eta} \text{qdim}(\lambda \hat{\otimes} \eta^{\dagger}) \eta = \text{qdim}(\lambda) \sum_{\eta} \text{qdim}(\eta) \eta = \text{qdim}(\lambda) \omega. \end{aligned}$$

(b) By the previous point, for any simple γ

$$\langle \lambda, \omega \rangle = \langle \lambda, \omega \hat{\otimes} \gamma \rangle / \text{qdim}(\gamma) = \langle \lambda, \gamma \rangle \langle \lambda, \omega \rangle / (\text{qdim}(\gamma) \text{qdim}(\lambda))$$

so either $\langle \lambda, \omega \rangle = 0$ or for every γ we have $\langle \lambda, \gamma \rangle = \text{qdim}(\lambda) \text{qdim}(\gamma)$. The second condition we have already noted is equivalent to the degeneracy of λ , so this only happens when $\lambda = \text{id}$ or $\lambda = \mu$. In both cases the formula follows immediately.

(c) Using the previous point

$$\begin{aligned} \langle \omega, \omega \rangle &= \sum_{\gamma} \text{qdim}(\gamma) \langle \gamma, \omega \rangle \\ &= \text{qdim}(\text{id}) \langle \text{id}, \omega \rangle + \text{qdim}(\mu) \langle \mu, \omega \rangle = 2 \text{qdim}(\omega). \end{aligned}$$

Of course $\text{qdim}(\omega) = \sum_{\gamma} \text{qdim}(\gamma)^2 > 0$.

□

Theorem 3. *Let L be an even link representing a three-manifold M with a compatible framing and let $I(L)$ be the invariant associated to the spin-modular category \mathfrak{C} acting on L with each component labeled by $\omega / \sqrt{2 \text{qdim}(\omega)}$. I.e., if L has components $1, \dots, n$ and $L_{\lambda_1, \dots, \lambda_n}$ is L with the n components labeled by simple objects $\lambda_1, \dots, \lambda_n$ respectively and F is the link invariant, then*

$$(1) \quad I(L) = \left(\frac{1}{\sqrt{2 \text{qdim}(\omega)}} \right)^n \sum_{\lambda_1, \dots, \lambda_n} \left(\prod_{i=1}^n \text{qdim}(\lambda_i) \right) F(L_{\lambda_1, \dots, \lambda_n})$$

where the sum is over isomorphism classes of simple objects and $\text{qdim}(\omega) = \sum_{\lambda} \text{qdim}(\lambda)^2$. Then $I(L)$ is invariant under the spin Kirby moves and therefore is an invariant of M and its framing.

Proof. Notice first that

$$\omega^{\dagger} = \sum_{\gamma} \text{qdim}(\gamma) \gamma^{\dagger} = \sum_{\gamma} \text{qdim}(\gamma^{\dagger}) \gamma^{\dagger} = \omega,$$

so $I(L)$ is unchanged by reversing the orientation of any component of L .

For invariance under Kirby move II as pictured in Figure 4, notice by Properties 3 and 6 of the link invariant the value of F on the link on the right is $\langle \omega, \omega \rangle / 2 \text{qdim}(\omega) = 1$ times that on the left, using Lemma 2(c).

For invariance under Kirby move I' as pictured in Figure 4, the argument is given pictorially in Figure 8, where the first equality is by Property 7 of the invariant, the second by Lemma 2(b), the third by the degeneracy of 0 and μ , the fourth by Lemma 2(b) and the fifth by Property 7 again. \square

$$\begin{aligned}
 F \left(\begin{array}{c} \text{---} \\ \text{---} \\ \omega \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right) &= \sum_i F \left(\begin{array}{c} \text{---} \\ \text{---} \\ \omega \\ \gamma_i \\ \text{---} \\ \text{---} \end{array} \right) = \\
 \sum_{i:\gamma_i=\text{id},\mu} F \left(\begin{array}{c} \text{---} \\ \text{---} \\ \gamma_i \\ \text{---} \\ \text{---} \end{array} \right) \text{qdim}(\omega) &= \sum_{i:\gamma_i=\text{id},\mu} F \left(\begin{array}{c} \text{---} \\ \text{---} \\ \gamma_i \\ \text{---} \\ \text{---} \end{array} \right) \text{qdim}(\omega) \\
 &= \sum_i F \left(\begin{array}{c} \text{---} \\ \text{---} \\ \omega \\ \gamma_i \\ \text{---} \\ \text{---} \end{array} \right) = F \left(\begin{array}{c} \text{---} \\ \text{---} \\ \omega \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right)
 \end{aligned}$$

FIGURE 8. Invariance under Kirby move I'

Proposition 3. *If \mathfrak{C} is a ribbon $*$ -category whose even degenerate objects form a cyclic group of invertibles and F is the associated link invariant then Equation (1) applied to the category \mathfrak{C} is an invariant of compatibly framed three-manifolds as above.*

Proof. The proof of Proposition 2 of [37] shows that if \mathfrak{C} is any ribbon $*$ -category and \mathfrak{C}' is the quotient as in [7, 28] of \mathfrak{C} by a cyclic group of invertible even degenerate objects, then the image of the expression $I(L)$ above under the functor from \mathfrak{C} to \mathfrak{C}' is the same expression in the image category (the statement of the proposition discusses only the case when the quotient is by the full set of degenerate objects, but the argument does not use this fact in any way). Thus if the full

set of even degenerate objects is a cyclic group of invertible elements and \mathfrak{C}' is therefore modular, the formula $I(L)$ gives an invariant of 2-framed three-manifolds as in [37], and if not then \mathfrak{C}' is spin-modular and $I(L)$ gives an invariant of compatibly framed three-manifolds as in the previous theorem. \square

Remark 5. Lemma 2(a) applied to a particular knot gives the invariance of ω under band connect sum of the unknot with that knot, and with a little more effort of any knot separated by a sphere with that knot. It does not appear to give the full invariance under move I (which would be conceptually superior to our indirect proof through move I') without substantially more effort. This effort would amount to switching focus from the space associated to the torus (roughly what we call V) to the space associated to the *punctured* torus. The appropriate setting for this would be the full axioms of an extended TQFT of Walker [40], whose generalization to the spin category offers a very interesting question.

Corollary 1. *Every closed subset Λ of the Weyl alcove, for every quantum group $U_q(\mathfrak{g})$ at every level k (i.e., for q an arbitrary root of unity) except possibly the exceptions listed below, yields a framed (or 2-framed) three-manifold invariant by the formula $I(L)$ above, where F is the standard quantum group link invariant. The possible exceptions are: \mathfrak{g} having Dynkin diagram D_{2n} and the set $\Gamma_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ of weights in the root lattice, where the group of degenerates is not cyclic, as well as at level $k = 2$ the exceptional sets for D_n and B_n discussed in [34], where the group associated to the subcategory of degenerates is not commutative.*

Proof. By [34], every closed subset of the Weyl alcove (closed meaning that the truncated tensor product of two elements of the set is the sum of elements of the set) yields a semisimple ribbon $*$ -subcategory of the standard ribbon category associated to the Weyl alcove whose degenerate objects are invertible and by [37] form (except for cases mentioned) a cyclic group. In [37] and [34] it is determined when these closed subsets include odd degenerate objects, and thus whether the formula $I(L)$ gives an invariant of 2-framed three-manifolds or compatibly framed three-manifolds. \square

Remark 6. Presumably Formula (1) is preserved by the quotient, and thus defines a 2-framed or compatibly framed three-manifold invariant, even when the subcategory of even degenerate objects is not generated by a cyclic group, but a proof is currently lacking.

2.3. Decomposition into prime invariants. In [37] it was found that many of the closed subsets of the Weyl alcove which give modular quotients and hence 2-framed three-manifold invariants could be factored as a product of others from the list. In this subsection we analyze the decomposition of subsets whose quotients are spin-modular.

Let \mathfrak{g} be a simple Lie algebra and G be the simply-connected compact Lie group with Lie algebra \mathfrak{g} , and consider the Weyl alcove at level k . If Z is a subgroup of the center $Z(G)$ of G , let Γ_Z be the intersection of the Weyl alcove with those weights in the Weyl chamber which are the highest weights of representations of G on which Z acts trivially. Let Δ_Z be the set of weights which are k times a fundamental weight λ whose inner product with every element of Γ_Z is an integer. In fact, there is a bijection ℓ from $Z(G)$ to a certain subset of the fundamental weights, and Δ_Z is the image $k\ell[Z]$ of Z , where k represents the linear map multiplication by k in the lattice. Then [37] shows that the sets Γ_Z and Δ_Z are closed.

Recall from [37] that if Γ is the set of isomorphism classes of simple objects of a ribbon category \mathfrak{C} (in particular if it is a closed subset of the Weyl alcove) then we say Γ is the product of two subsets Γ' and Γ'' if

1. the intersection $\Gamma' \cap \Gamma''$ consists of even degenerate objects,
2. the product $\hat{\otimes}$ of any element of Γ' with an element of Γ'' is simple (i.e. is an element of Γ),
3. every element of Γ is a product of an element of Γ' and Γ'' and
4. if $\lambda' \in \Gamma'$ and $\lambda'' \in \Gamma''$ then $C_{\lambda' \hat{\otimes} \lambda''} = C_{\lambda'} C_{\lambda''}$.

Proposition 4. *If Γ is the product of Γ' and Γ'' then*

$$(2) \quad I_{\Gamma}(L) = I_{\Gamma'}(L)I_{\Gamma''}(L)$$

where I is the invariant of the previous subsection computed in the categories associated to Γ , Γ' , and Γ'' respectively.

Proof. As in the previous subsection let V_{Γ} be the formal vector space spanned by isomorphism classes of simple objects in \mathfrak{C} , and let $V_{\Gamma'}$ and $V_{\Gamma''}$ be the corresponding vector spaces for \mathfrak{C}' and \mathfrak{C}'' . Of course the truncated tensor product gives an algebra homomorphism $\phi : V_{\Gamma'} \otimes V_{\Gamma''} \rightarrow V_{\Gamma}$ which by point 3 in the definition above is onto. It is shown in [37] that the link invariant with a component labeled by $\phi(a \otimes b)$ is the product of the link invariants with components labeled by a and b respectively. From this we see that $\phi(\omega' \otimes \omega'')$ has the property that $\phi(\omega' \otimes \omega'')v = \phi(\omega' \otimes \omega'') \text{qdim}(v)$, because $\omega' \otimes \omega''$ has this property

in $V_{\Gamma'} \otimes V_{\Gamma''}$. Thus

$$\begin{aligned} \phi(\omega' \otimes \omega'') \operatorname{qdim}(\omega) &= \phi(\omega' \otimes \omega'')\omega \\ &= \operatorname{qdim}(\phi(\omega' \otimes \omega''))\omega = \operatorname{qdim}(\omega') \operatorname{qdim}(\omega'')\omega, \end{aligned}$$

the last equality being a consequence of the behavior of the invariant under ϕ . Thus $\phi(\omega' \otimes \omega'')$ and ω are nonzero multiples of each other, since $\operatorname{qdim}(\omega)$, $\operatorname{qdim}(\omega')$, and $\operatorname{qdim}(\omega'')$ are all nonzero. Since $I(L)$ is easily seen to be unchanged if we replace ω by a nonzero multiple, the result follows from the invariant's behavior under ϕ . \square

The following is proven in [37].

Proposition 5. *Suppose $Z \subset Z'$ are subgroups of the center of G , Γ' is the closed subset generated by $\Gamma_{Z'}$ and Δ_Z , and $Z_0 = Z \cap (k \circ \ell)^{-1}[\Gamma_{Z'}]$. Then $\Gamma' \subset \Gamma_{Z_0}$ is of the form $\Gamma' = \Gamma_Y$ for some Y and $\Delta_{Z_0} = \Delta_Z \cap \Gamma_{Z'}$ consists of degenerate invertible objects for Γ' . If all of Δ_{Z_0} is even then Γ' is the product of $\Gamma_{Z'}$ and Δ_Z . These are the only cases in which Γ decomposes into a product, apart from D_{2n} . \square*

We shall be particularly interested in theories coming from closed subsets Γ_Z with the property that $\Delta_Z \subset \Gamma_Z$. The following proposition shows that all other closed sets Γ_Z appear as factors of these, and the next proposition gives conditions for when a closed set has this property.

Proposition 6. *If $\mathfrak{g} \neq D_{2n}$, every $\Gamma_{Z'}$ is a factor of a Γ_{Z_0} with the property that $\Delta_{Z_0} \subset \Gamma_{Z_0}$.*

Proof. Given $\Gamma_{Z'}$, let Z be Z' if all λ in $\Delta_{Z'} \cap \Gamma_{Z'}$ satisfy $C_\lambda = 1$. If $\Delta_{Z'} \cap \Gamma_{Z'}$ contains any elements with $C_\lambda = -1$, then those with $C_\lambda = 1$ form an index two subgroup, and since $\Delta_{Z'}$ is cyclic, $\Delta_{Z'}$ too must have an index two subgroup whose intersection with $\Gamma_{Z'}$ has only objects with $C_\lambda = 1$. In that case let Z be the corresponding index two subgroup of Z' . Then $\Delta_{Z'}$ is generated by Δ_Z and any element of $\Delta_{Z'} \cap \Gamma_{Z'}$ with $C_\lambda = -1$. Thus in particular $\Delta_{Z'}$ is generated by Δ_Z and elements of $\Gamma_{Z'}$. So whatever Z is $\Gamma_{Z'}$ and Δ_Z generate the same set Γ' generated by $\Gamma_{Z'}$ and $\Delta_{Z'}$. The previous proposition shows $\Gamma_{Z'}$ and Δ_Z generate Γ_{Z_0} and that $\Delta_{Z_0} = \Delta_Z \cap \Gamma_{Z'}$ consists of even degenerate objects for Γ_{Z_0} , so Γ_{Z_0} is the product of $\Gamma_{Z'}$ and Δ_Z . \square

Proposition 7. *Γ_Z contains Δ_Z as degenerate objects if and only if*

- (a) *$k(\ell(z), \ell(z))$ is an even integer for all $z \in Z$, in which case Γ_Z contains only even degenerate objects or*

(b) $k(\ell(z), \ell(z))$ is an integer for all $z \in Z$, with at least one of these integers odd, in which case Γ_Z contains an odd degenerate object and Z contains an index 2 subgroup.

Proof. Recall from [37] $\lambda \in \Gamma_Z$ if and only if $(\lambda, \ell(z)) \in \mathbb{Z}$ for all $z \in Z$. since Z is cyclic, this is equivalent to $(\lambda, \ell(z)) \in \mathbb{Z}$ for z a generator. So $\Delta_Z \in \Gamma_Z$ is equivalent to $k(\ell(z'), \ell(z)) \in \mathbb{Z}$ for all $z' \in Z$ and for some generator $z \in Z$, which is to say $k(\ell(z), \ell(z)) \in \mathbb{Z}$ for some generator $z \in Z$, or equivalently for all $z \in Z$.

Now it is shown in [37] that

$$R_{\lambda, k\ell(z)} R_{k\ell(z), \lambda}^{-1} = e^{2\pi i k(\ell(z), \lambda)},$$

so $k\ell(z)$ with $z \in Z$ is always degenerate for Γ_Z if it is in Γ_Z . thus we need only check the additional assertions in (a) and (b). Again from [37], $C_{k\ell(z)} = \exp(\pi i k(\ell(z), \ell(z)))$, so this is one if and only if $k(\ell(z), \ell(z))$ is even. Finally, notice the set of $z \in Z$ such that $k(\ell(z), \ell(z))$ is even forms a subgroup. If it is proper it has index two, because the product of two elements not in this subgroup is in this subgroup. \square

Finally, we observe that when Δ_Z is cyclic and contains an odd degenerate object, the framed invariant I_{Δ_Z} that results is actually an ordinary 2-framed invariant. In fact it is the invariant Murakami, Ohtsuki and Okada associate to the quotient of Z by the entire group of degenerates [29]. Specifically, let λ be a generator of Δ_Z , and suppose N is the least such that λ^N is a degenerate object. Recalling the above formulas for $C_{k\ell(z)}$ and $R_{k\ell(z), \gamma} R_{\gamma, k\ell(z)}^{-1} = C_{k\ell(z) \otimes \gamma} C_{k\ell(z)}^{-1} C_{\gamma}^{-1}$, to say that λ^n is degenerate is to say that $C_{\lambda^{n+1}} = C_{\lambda^n} C_{\lambda}$, since it suffices to check the degeneracy condition against a generator. Now $\lambda = k\ell(z)$ and $\lambda^n = k\ell(z^n)$ for some $z \in Z$, so if we let $r = \exp(\pi i (k\ell(z), \ell(z)))$ then $C_{\lambda} = r$ and

$$R_{\lambda^n, \lambda} R_{\lambda, \lambda^n} = e^{2\pi i k(\ell(z), \ell(z^n))} = r^{2n}.$$

So by induction $C_{\lambda^n} = \lambda^{n^2}$. Thus N is the least such that λ^N is degenerate if and only if N is the least N such that $r^{2N} = 1$.

If Δ_Z contains an odd degenerate object then λ^N is odd, and thus $C_{\lambda^N} = r^{N^2} = -1$. We conclude that N is odd and r is a primitive $2N$ th root of unity.

Now quotient by the even degenerate objects so that Z becomes \mathbb{Z}_{2N} and we can identify the simple object λ^m with the number $m \in \mathbb{Z}_{2N}$. We claim that the invariant of a link with n components labeled by the entries in $\vec{l} = (l_1, \dots, l_n) \in (\mathbb{Z}_{2N})^n$ is $r^{\vec{l} A \vec{l}}$, where A is the linking matrix of the link. To see this notice the formula agrees with the invariant for

a link composed of n unlinked but possibly framed unknots, and that the invariant and the proposed formula both change by r^{2pq} when a crossing between components labeled by p and q respectively switches parity.

Thus the compatibly framed invariant is

$$I_{\Delta_Z}(L) = (2\sqrt{N})^{-n} \sum_{\vec{l} \in (\mathbb{Z}_{2N})^n} r^{\vec{l} A \vec{l}}.$$

But if A is an even matrix then $\vec{l} A \vec{l}$ is always even, so we may replace r by the primitive N th root of unity $-r$ without changing the value. In that case the contribution from any \vec{l} is the same as from $\vec{l} + (0, 0, \dots, 0, N, 0, \dots, 0)$ and thus we may take the sum over \mathbb{Z}_N at the cost of multiplying by 2^n , and thus get

$$I_{\Delta_Z}(L) = (\sqrt{N})^{-n} \sum_{\vec{l} \in (\mathbb{Z}_N)^n} (-r)^{\vec{l} A \vec{l}}.$$

Now notice this is exactly the invariant Murakami, Ohtsuki and Okada call $Z(-r, N)$, which is a 2-framed three-manifold invariant. Thus it does not depend on the spin structure.

2.4. Relationship to geometry and physics. In [10], Dijkgraaf and Witten discuss under what circumstances one expects a TQFT and three-manifold invariant to arise from the Chern-Simons field theory of a (possibly nonsimply-connected) compact simple Lie group G . Their approach is to define the Chern-Simons functional when the principal G -bundle over the three-manifold is not trivial in terms of a cohomology class in $H^4(BG, \mathbb{Z})$. The action is computed by choosing a four-manifold and principal bundle (or more generally a homology class in $H_4(BG, \mathbb{Z})$) bounded by the three-manifold and principal bundle and pairing the cohomology class with the fundamental class of the four-manifold. This result must always be an integer when the three-manifold is trivial in order for the path integral to be well-defined, which is why $H^4(BG)$ must be taken with integer coefficients and the reason for the integrality conditions on k derived by Dijkgraaf and Witten.

Dijkgraaf and Witten add an intriguing point. If the group is such that the generating class of $H^4(BG, \mathbb{Z})$ (Recall $H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$ for every compact simple group G except the one associated to the Dynkin diagram D_{2n} with trivial center) when integrated against the fundamental class of a *spin* four-manifold is even, then half-integer multiples of the generator (which we may think of as $H^4(BG)$ classes with half-integer coefficients) would give a well-defined action if the four-manifold

is forced to be spin. Thus in these cases we expect half-integer cohomology classes to give spin TQFTs (as in the ordinary case the theory still depends on the signature of the extending four-manifold, i.e., the 2-framing, so really we expect compatibly framed TQFTs). Dijkgraaf and Witten show that $SO(3)$ in particular has this property and discuss the expected spin TQFT in this case (which should occur when the $SU(2)$ level is 2 modulo 4).

In fact the property is quite general and occurs exactly in the situation covered by Proposition 7.

Proposition 8. *Suppose G is a compact Lie group such that $H^4(BG, \mathbb{Z}) = \mathbb{Z}$ and $\pi_1(G)$ has an index two subgroup (i.e. g admits a double cover). Then the generating class of $H^4(BG, \mathbb{Z})$ pulled back via a G -bundle on a spin four-manifold has even integral.*

Proof. Since $H^4(BG, \mathbb{Z}) = \mathbb{Z}$ it follows $H^4(BG, \mathbb{Z}_2) = \mathbb{Z}_2$. We claim there is a $w \in H^2(BG, \mathbb{Z}_2)$ such that $w \wedge w$ is nontrivial and hence is equal to the unique nontrivial element of $H^4(BG, \mathbb{Z}/2)$. Specifically, w is the obstruction to lifting a G bundle to a \tilde{G} bundle, \tilde{G} being its double cover, which necessarily exists because of the condition on $\pi_1(G)$.

To see that $w \wedge w$ is nontrivial, consider $\mathbb{C}P^2$, the four-manifold constructed by the surgery process of Section 2 from the +1-framed unknot. More precisely, it is formed by attaching a single 2-handle to a 0-handle along the framed unknot and then attaching a 4-handle to the resulting S^3 boundary. We construct a G -bundle over $\mathbb{C}P^2$ as follows. Attach the trivial bundle over the 2-handle to the trivial bundle over the 0-handle via an overlap map on the boundary which is homotopic to an element of $\pi_1(G)$ not in the index two subgroup. Extend this bundle over the 4-handle. Of course this bundle does not lift to \tilde{G} , so the image \tilde{w} of w in $H^2(\mathbb{C}P^2, \mathbb{Z}_2) = \mathbb{Z}_2$ is nontrivial. The image of $w \wedge w$ is $\tilde{w} \wedge \tilde{w}$ which is nontrivial because the intersection pairing given by the 1×1 identity linking matrix is nondegenerate.

On four-manifolds with even forms (which includes all spin four-manifolds) $\int_M w \wedge w = 0 \pmod{2}$, so the integral of every class of $H^4(BG, \mathbb{Z})$ against such four-manifolds is even. \square

Corollary 2. *The techniques of Dijkgraaf and Witten predict a spin (really a compatibly framed) Chern-Simons theory at the levels and groups given in Proposition 7(b).*

Proof. Dijkgraaf and Witten's techniques, in conjunction with the previous proposition, predict a spin TQFT associated to the Lie group

G/Z with half-integer levels, where G is a simply-connected group and Z is a subgroup of the center containing an index two subgroup. Now integer levels of G/Z correspond as levels of G to integer multiples of N , N being the least integer such that $N(\ell(z), \ell(z))$ is an even integer for all $z \in Z$. So half-integer levels correspond to odd multiples of $N/2$.

Thus Dijkgraaf and Witten predict a spin TQFT for G/Z at level k if $k(\ell(z), \ell(z))$ is an integer for all $z \in Z$ (because it is a half-integer multiple of an even number) but at least one of these numbers is odd (otherwise $N(\ell(z), \ell(z))$ is a multiple of 4 for all $z \in Z$, and N would not be the least such meeting the defining condition). □

From Dijkgraaf and Witten's point of view we should interpret these spin Chern-Simons theory as theories we compute on spin four-manifolds, and perhaps even as invariants of spin four-manifolds which happen to depend only on their boundary (and signature). By defining our invariants in terms of surgery, we have partially modeled this feature. By assigning numbers to even links, we are really assigning numbers to certain spin four-manifolds, and discovering the fact that the numbers depend only on the spin boundary and signatures. Of course our numbers are only assigned to spin four-manifolds which admit a handle decomposition as one 0-handle and some 2-handles. It is an interesting question whether from a spin-modular category one can naturally associate to *every* spin four-manifold an invariant which reduces to this one when the four-manifold admits a handle decomposition as described. Presumably this would involve assigning some sort of a label to link components representing 1-handles (Kirby's dotted circles [22]). This question may shed light on efforts to construct interesting four-manifold invariants from algebraic structures related to quantum groups.

2.5. Identification with spin invariants of Kirby and Melvin. A special case of the construction of the preceding subsections is the group $\mathrm{SO}(3)$ which is $\mathrm{SU}(2)/\mathbb{Z}_2$. Here the center contains a single nontrivial element z and $\ell(z)$ is the unique fundamental weight with $(\ell(z), \ell(z)) = 1/2$ and thus $N = 4$. So we expect spin $\mathrm{SO}(3)$ invariants at $\mathrm{SU}(2)$ levels which are 2 modulo 4. These are exactly the levels at which Kirby and Melvin [21] construct invariants of spin three-manifolds from the representation theory of quantum \mathfrak{su}_2 (see also Blanchet [5], as well as [6], where Blanchet and Masbaum define a spin TQFT giving this invariant). We will show that Kirby and Melvin's invariant is exactly the $\mathrm{SO}(3)$ invariant, and in fact that every spin invariant of the previous

subsection can be computed in a manner analogous to theirs using the subset of representations associated to the double cover.

Recall that Kirby and Melvin present a framed three-manifold by a link L with a *characteristic sublink* C such that for each component a of L

$$\sum_{c \in C} a \cdot c = a \cdot a \pmod{2}$$

where the dot represents the linking number between the two components (or in the case of $a \cdot a$, the framing). This corresponds to the three-manifold obtained by surgery on the link L , with the unique spin structure which when restricted to $S^3 - L$, extends over the components of $L - C$ but not over the components of C . We describe a generalization of their invariant and show that it gives exactly the spin invariants we constructed in the previous subsections.

Let $\Delta_Z \subset \Gamma_Z$ be such that Γ_Z contains odd degenerate objects. Then as in Proposition 7, the subset $Z_0 = \{z \in Z : k(\ell(z), \ell(z)) \in 2\mathbb{Z}\}$ is an index two subgroup, and therefore $\Gamma = \Gamma_{Z_0}$ has Γ_Z as a subset and contains only even degenerate objects.

Let

$$\begin{aligned} \omega &= \sum_{\gamma \in \Gamma} \text{qdim}(\gamma)\gamma \\ \omega_0 &= \sum_{\gamma \in \Gamma_Z} \text{qdim}(\gamma)\gamma \\ \omega_1 &= \omega - \omega_0. \end{aligned}$$

Proposition 9. *If L is a link with characteristic sublink C , let $F(L, C)$ be the invariant (in Γ) of L with every component of C labeled by ω_1 and every other component of L labeled by ω_0 , let U_+ and U_- be the invariants of the respectively $+1$ and -1 framed unknots labeled by ω_1 , and let n be the number of components of L . Then*

$$(3) \quad J(L, C) = F(L, C)/(U_+U_-)^{n/2}$$

is an invariant of the framed (i.e. 2-framed and spin) manifold determined by (L, C) and

$$(4) \quad J'(L, C) = \left(\frac{U_-}{U_+}\right)^{\sigma/2} J(L, C)$$

is an invariant of the ordinary spin manifold determined by (L, C) .

Proof. We shall confirm invariance of the second quantity, that of the first follows. Invariance under orientation reversal is clear.

According to Kirby and Melvin an invariant of a link with a characteristic sublink is an invariant of the spin manifolds if it is unchanged by the following move: add a ± 1 framed unknot to the link (possibly linking with other components), apply a positive or negative full twist to the disk it bounds (so as to change the linking matrix of the link) and add it to the characteristic sublink if and only if the sum of its linking numbers with the existing characteristic sublink is even.

Notice first that as argued earlier the formula $J'(L, C)$ applied to a link gives the same answer as the same formula interpreted in the quotient of Γ by Z_0 , so we may assume that Γ is modular and that Γ_Z is spin modular. Let us call the unique odd degenerate object in Γ_Z μ .

Observe first that if λ is a simple object of $\Gamma - \Gamma_Z$, then $S_{\lambda, \mu} = -\text{qdim}(\lambda)$. To see this, note that since μ^2 is the trivial object and $\text{qdim}(\lambda) = S_{\lambda, \mu \hat{\otimes} \mu} = S_{\lambda, \mu}^2 / \text{qdim}(\lambda)$ we have that $S_{\lambda, \mu} = \pm \text{qdim}(\lambda)$. But supposing $S_{\lambda, \mu} = \text{qdim}(\lambda)$, then if γ is an other simple object in Γ but not in Γ_Z , then $\lambda \hat{\otimes} \gamma \in \Gamma_Z$, so that $\text{qdim}(\lambda) \text{qdim}(\gamma) = S_{\lambda \hat{\otimes} \gamma, \mu} = S_{\lambda, \mu} S_{\gamma, \mu} = \text{qdim}(\lambda) S_{\gamma, \mu}$ and we conclude $S_{\gamma, \mu} = \text{qdim}(\gamma)$. But this is certainly true for $\gamma \in \Gamma_Z$, so μ is degenerate for Γ . This is a contradiction so $S_{\lambda, \mu} = -\text{qdim}(\lambda)$.

Consider the result of Kirby and Melvin's move, and suppose first that the unknot is to be added to the characteristic sublink, because its linking number with the old sublink is even. The invariant $F(L, C)$ is a sum over labelings of the components of L , with those in C labeled by elements of $\Gamma - \Gamma_Z$ and those not in C labeled by elements of Γ_Z . Choose such a labeling, and consider the invariant of this labeled link with a particular label κ on the new unknot. The condition on the linking number means that the new unknot surrounds a collection of strands an even number of which have labels not in Γ_Z , and therefore the tensor product of all these labels is a sum of labels in Γ_Z (here we use Property 7 of the ribbon category). If κ is in Γ_Z , then $\kappa \hat{\otimes} \mu$ is a distinct label with $C_{\kappa \hat{\otimes} \mu} = -C_\kappa$ and $S_{\gamma, \kappa} = S_{\gamma, \kappa \hat{\otimes} \mu}$ for $\gamma \in \Gamma_Z$ and we see that labeling the new unknot by κ versus $\kappa \hat{\otimes} \mu$ contributes the same amount with opposite sign to the computation of the total invariant. Thus labeling the new unknot by ω_0 would give a total invariant of 0, so that in the computation of $J'(L, C)$ we might as well replace the label ω_1 on the new unknot with ω . The same argument applies to U_+ and U_- , so we see that in this case the invariance of $J'(L, C)$ under this move is equivalent to the invariance of the ordinary manifold invariant of Γ under this move.

Now suppose that the new unknot is not to be added to the characteristic link, because its linking number with the characteristic link is

odd. The same argument as above shows the truncated tensor product of the labels of the strands going through the new unknot is a sum of elements of $\Gamma - \Gamma_Z$. If κ is a label for the new unknot which is not in Γ_Z and we compare the effect on the invariant of replacing κ by $\kappa \hat{\otimes} \mu$, we see that $C_{\kappa \hat{\otimes} \mu} = C_\kappa$, but now an unknot labeled by μ surrounding a sum of labels not in Γ_Z contributes -1 , so again κ and $\kappa \hat{\otimes} \mu$ contribute opposite amounts to the sum (in this case they may not be distinct, but then κ contributes zero) and thus labeling the unknot by ω_1 results in a total invariant of zero. Once again we may as well replace the label of ω_0 with ω , and again the results follows from the invariance of the standard invariant under the move. \square

In the case $\mathfrak{g} = \mathfrak{su}_2$, $k \equiv 2 \pmod{4}$, and $Z = \mathbb{Z}_2$, we have that Γ is the whole set of representations, Γ_Z is the set of integer spin representations, and our formula reduces exactly to Kirby and Melvin's formula (they sum only over half the representations, but using their symmetry principle, this is equivalent to summing over all the representations, as they note).

That this invariant is the one we already constructed is now obvious by taking the characteristic sublink to be empty.

Proposition 10. *The invariant $J(L, C)$ when applied to a compatibly framed three-manifold, gives the same result as the invariant I associated to Γ_Z .*

Proof. If we present the compatibly framed manifold by an even link L , then notice that the empty link is a characteristic sublink, and (L, \emptyset) is the Kirby-Melvin presentation of this 2-framed spin three-manifold. Of course $F(L, \emptyset) = F(L)$ as defined in the second subsection of this section, and thus the invariants are equal as long as the normalizations are equal, that is if $U_+U_- = F(H)$, with H the Hopf link labeled by ω_0 . But of course the Hopf link represents S^3 with the standard spin structure and 2-framing, so $J(H, \emptyset) = 1$, which means $F(H) = U_+U_-$. \square

Remark 7. Kirby and Melvin's argument that the sum of the Γ_Z invariant over all possible spin structures on a given manifold adds up to the Γ invariant of the manifold goes through in the general case by an analogous argument.

Remark 8. In the Kirby-Melvin formulation of the invariant, any 2-framed spin three-manifold can be represented by a link and characteristic sublink, and thus we get an invariant of 2-framed spin-manifolds, without the compatibility constraint. This extension of our framed

invariant amounts to a canonical choice of a sixteenth root of the invariant of the Kummer surface, and thus the additional information it detects is at most Rohlin's invariant. Whether there is a canonical way to do this in a more general ribbon $*$ -category (i.e., a category that does not already come embedded with index 2 in a category without odd degenerate objects), or in the situation of the next section, is an open question.

Remark 9. If we consider \mathfrak{su}_2 at level $k = 2$, then $\Delta_{\mathbb{Z}_2} = \Gamma_{\mathbb{Z}_2}$ is a closed subset consisting of the trivial object and an odd degenerate object. The invariant $I_{\Delta_{\mathbb{Z}_2}}$ is the Murakami et al's invariant $Z(1, 1)$, which is completely trivial (it assigns 1 to all 2-framed manifolds). However, the Kirby-Melvin extension is nontrivial, as it depends on Rohlin's invariant (in fact it is $\exp(-3\pi i \mu(M)/8)$).

3. HENNINGS TYPE SPIN INVARIANTS

3.1. Even link invariants from quasitriangular Hopf algebras.

Recall that a *quasitriangular Hopf algebra* is a Hopf algebra

$$(H, \cdot, 1, \Delta, \epsilon, S)$$

(so that H is an algebra with multiplication \cdot and unit 1, the dual space H^\dagger is an algebra with multiplication Δ^\dagger and unit ϵ , $\Delta: H \rightarrow H \otimes H$ is a homomorphism, and $S: H \rightarrow H$ is an antihomomorphism satisfying $x^{(1)}S(x^{(2)}) = S(x^{(1)}) \otimes x^{(2)} = \epsilon(x)1$, where $\Delta(x) = x^{(1)} \otimes x^{(2)}$ is Sweedler's index-saving notation) together with an element $R = \sum_i a_i \otimes b_i \in H \otimes H$ such that

$$(5) \quad \Delta^{\text{op}}(x)R = \sum_i (x^{(2)}a_i \otimes x^{(1)}b_i) = \sum_i (a_i x^{(1)} \otimes b_i x^{(2)}) = R\Delta(x)$$

$$(6) \quad (\Delta \otimes \text{id}_H)(R) = \sum_i (a_i^{(1)} \otimes a_i^{(2)} \otimes b_i) \\ = \sum_{i,j} (a_i \otimes a_j \otimes b_i b_j) = R_{1,2}R_{2,3}$$

$$(7) \quad (\text{id}_H \otimes \Delta)(R) = \sum_i a_i \otimes b_i^{(1)} \otimes b_i^{(2)} \\ = \sum_{i,j} (a_i a_j \otimes b_j \otimes b_i) = R_{1,3}R_{1,2}.$$

From these relations follow all sorts of useful facts, including

$$(8) \quad R_{1,2}R_{1,3}R_{2,3} = \sum_{i,j,k} (a_i a_j \otimes b_i a_k \otimes b_j b_k) \\ = \sum_{i,j,k} (a_j a_k \otimes a_i b_k \otimes b_i b_j) = R_{2,3}R_{1,3}R_{1,2}$$

$$(9) \quad (\epsilon \otimes \text{id}_H)(R) = \sum_i \epsilon(a_i) b_i = 1 = \sum_i \epsilon(b_i) a_i = (\text{id}_H \otimes \epsilon)(R)$$

$$(10) \quad (S \otimes \text{id}_H)(R) = \sum_i S(a_i) \otimes b_i = \\ R^{-1} = \sum_i a_i \otimes S^{-1}(b_i) = (\text{id}_H \otimes S^{-1})(R)$$

$$(11) \quad (S \otimes S)(R) = \sum_i S(a_i) \otimes S(b_i) = \sum_i a_i \otimes b_i = R.$$

If H also possesses an element $g \in H$ such that $\Delta g = g \otimes g$, $\epsilon(g) = 1$, $g^{-1} u g^{-1} = S(u)$ and $S^2(x) = g x g^{-1}$ for all $x \in H$, where $u = \sum_i S(b_i) a_i$, then H is called a *ribbon Hopf algebra*. Its representation theory forms a ribbon category, and associated to it by the recipe of Ohtsuki [30] or Kauffman and Radford [16] is a numerical invariant of framed links with components labeled by *quantum characters*. Here a quantum character of H is a functional $f \in H^\dagger$ such that $f(\text{Ad}_x(y)) = \epsilon(x) f(y)$, where $\text{Ad}_x(y) = x^{(1)} y S(x^{(2)})$. Equivalently (according to Drinfel'd [11]), a quantum character is a functional f such that $f(xy) = f(y S^2(x))$ for all $x, y \in H$.

Here we associate to a finite-dimensional quasitriangular (but not necessarily ribbon) Hopf algebra H an invariant of *even* links with components labeled by quantum characters. The invariant is defined precisely in analogy with the definition for ribbon categories (in fact, if the quasitriangular Hopf algebra is extended in the usual way of Reshetikhin and Turaev [32] to a ribbon Hopf algebra, the invariants agree on even links), and we will imitate the construction of [16] closely.

Let L be an even link with a quantum character λ_j associated to each component C_j of L . Choose a presentation of L with every component having winding number one, and also with a height function. We associate to the projection a collection of *decorated projections*, with Hopf algebra elements assigned to various noncritical points on each component as follows. For each crossing, choose one value of the index i in $R = \sum_i a_i \otimes b_i$ (we call such a choice for all crossings a *state*), and label points near the crossing as shown in Figure 9, where

the vertical in the figure is the height function. Now choose a base point on each component which is not a crossing or a critical point of the height function. Let us call the direction along the component which at the basepoint is of increasing height function the *basepoint direction*. To each noncritical point of each component, associate an *integer rotation number* by counting all the critical points visited when traveling from the base point to that point in the basepoint direction, counting each critical point which rotates clockwise as -1 and each which rotates counterclockwise as $+1$. That is, the nearest integer to the actual rotation from the basepoint to that point divided by π .

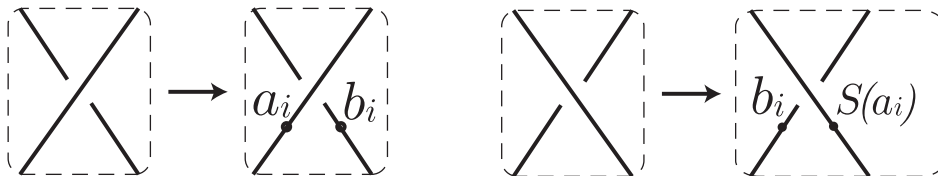


FIGURE 9. Labels assigned to crossings

For each state, assign a number to each component of the associated decorated link projection as follows. Travel from the base point once around the component in the basepoint direction. Form the product of all the decorated Hopf algebra elements on the component, left to right in the order of visitation, each acted on by S raised to the integer rotation number. The number assigned to the component is λ_i of this product if the basepoint direction coincides with the orientation of the component and is λ_i of S of the product if they are in opposite directions. The number assigned to the state is the product of the numbers assigned to the components, and finally the number assigned to the projection is the sum of over all states of the numbers assigned to the states.

Given a state, or any decorated link diagram, it is possible to compute the number assigned to that state ‘in stages.’ Specifically, a *fragment* of a link projection is the intersection of the projection with a disk whose boundary intersects the projection transversely and not at double points or critical points. Choose for each strand of the fragment a basepoint (several strands may belong to the same component of the entire link). Now each decorated point in the fragment has associated to it an integer rotation relative to the basepoint of its strand. Assign to each strand the product of $S^{n_i}(x_i)$, where x_i is a decoration, n_i is the integer rotation number, and the product is over all decorations. The order of the product comes from traversing the strand in the basepoint

direction, traveling from one endpoint to the other for open strands, and from the basepoint once around for closed strands.

The key observation is that if we erase all decorations in the fragment and replace them with the chosen basepoints decorated with the assigned products as above, the number associated to the entire state is unchanged.

In particular, notice that if we define the set of states of a fragment to be the set of all possible assignments of index values to the crossings in the fragment, then the set of states of the projection is the product of the set of states of the fragment and the set of states of its complement (which is also a fragment). So the number associated to the entire projection, which is a sum over states, can be found by summing over all pairs of states a number computed by evaluating the state on the fragment and on its complement and combining appropriately.

What's more, if a fragment of a projection is replaced with a perhaps topologically distinct fragment, but such that the strands connect the same pairs of boundary points, the rotation numbers of the boundary points from the basepoint are the same, and the sum of the decorations of the basepoints, viewed as an element of $H^{\otimes n}$ where n is the number of open strands, is the same, then the projection with the replaced fragment will yield the same number as the original projection. The proof of the following proposition offers a concrete illustration of this observation, where it is the key to proving invariance under the regular isotopy moves.

Proposition 11. *The number computed above depends neither on the specific projection nor on the choice of basepoints, but only on the even link and quantum characters. We call this invariant quantity $K(L)$, or $K_{\lambda_1, \dots, \lambda_n}(L)$ when the labels require explicit mention.*

Proof. For invariance under choice of basepoint, it suffices to check the quantity is unchanged if one basepoint is moved past a crossing or a critical point. This fact is generally true for decorated projections and does not rely on special properties of the R -matrix.

To see that it is unchanged when a basepoint is moved past a crossing or any decorating Hopf algebra element, consider for a given state and component, the product $PS^{\pm 2}(x)$, where x is the last decoration encountered in the traversal about the component and P is the product of the rest of the decorating elements with appropriate powers of S . The ± 2 represents the integer rotation number of the point decorated by x , which because of the winding number condition on the projection is 2 if the direction of traversal agrees with the orientation and -2 if it disagrees. In either case the defining property of the quantum

character gives

$$\begin{aligned}\lambda_i(P S^2(x)) &= \lambda_i(xP), \\ \lambda_i(S(P S^{-2}(x))) &= \lambda_i(S^{-1}(x)S(P)) = \lambda_i(S(P)S(x)) = \lambda_i(S(xP)),\end{aligned}$$

so the computed quantity is the same as for the case where the base point is just below instead of just above the decorated point.

To see that it is unchanged when the base point is moved past a critical point, consider a base point at which the orientation of the component is upwards, which is to be moved past a minimum of the height function configured so that the orientation rotates clockwise around it (the other cases all work similarly). Viewing the rest of the link as a fragment with the fragment basepoint just above the basepoint for the component in the link and decorated by x , the value of the decorated link projection before moving the basepoint is $\lambda_i(x)$, and after moving it is $\lambda_i(S(S^{-1}(x))) = \lambda_i(x)$.

To see that the invariant is independent of the choice of projection, notice by Proposition 1 we may check that it does not change under regular isotopy and the height function moves.

The first regular isotopy move reduces to the equation

$$\sum_{i,j} S(a_i)a_j \otimes b_i b_j = 1 \otimes 1 = \sum_{i,j} a_i S(a_j) \otimes b_i b_j$$

which is a restatement of Equation (10). The second isotopy move is exactly the Yang-Baxter equation (8). The height function moves are immediate from the definition. \square

Remark 10. In the presence of a ribbon structure the trace with respect to a representation V of the Hopf algebra can be made into a quantum character by adding the charmed element, $\lambda_V(\cdot) = \text{tr}(g^{-1}\cdot)$. The link invariant described in this section would then correspond to the usual Reshetikhin and Turaev link invariant associated to V [32]. For a nonsemisimple Hopf algebra there are typically other quantum characters. But even in the absence of the ribbon structure, for every representation V we can always choose an intertwiner from $V \otimes V^\dagger$ to the trivial representation (if V is irreducible, it is unique up to scale) and viewing elements of the Hopf algebra via the representation as elements of $V \otimes V^\dagger$, we get a functional on the Hopf algebra which proves to be a quantum character. Thus even in the merely quasitriangular situation representations give (even) link invariants. We expect that there is a categorical structure, analogous to but weaker than the notion of ribbon category, which axiomatizes the structure that allows the category of representations of such a Hopf algebra to give even link

invariants. It seems plausible that the quotient of a ribbon category containing odd degenerate objects by the full set of degenerate objects, which does not make sense as a ribbon category, would be well-defined as a category of this sort. We conjecture further that associated to the half-integer level Chern-Simons theories of the previous section there are quasitriangular but not ribbon Hopf algebras such that an appropriate truncation of their representation theory (corresponding to the truncated tensor product construction for ordinary quantum groups) yields the spin Chern-Simons theories at those levels.

A few important facts about K will be used in the next section, all following easily from the form of the computation:

1. $K(L_1 \cup L_2) = K(L_1)K(L_2)$, where $L_1 \cup L_2$, the distant union of the labeled links L_1 and L_2 , is the link formed by embedding each into S^3 so that they are separated by a sphere.
2. Reversing the orientation of a component corresponds to composing the quantum character labeling that component with S .

3.2. Integrals and the three-manifold invariant. This section relies heavily on some general results about finite-dimensional Hopf algebra. A reference that contains everything we need is Radford [31].

Recall that a left (respectively right) integral in the dual of a Hopf algebra H is a functional $\lambda \in H^\dagger$ such that $\gamma\lambda = \gamma(1)\lambda$ (respectively $\lambda\gamma = \gamma(1)\lambda$) for all $\gamma \in H^\dagger$. We say H is unimodular if there is a λ which is a left and right integral simultaneously, in which case it is unique up to scale. If λ is a left and right integral, then by [31, Theorem 3], $\lambda(ab) = \lambda(bS^2(a))$ for all $a, b \in H$, so that λ is a quantum character. Also by Proposition 3 of the same article $\lambda \circ S = \lambda$, so that (assuming now that H is quasitriangular and thus determines a link invariant as described in the previous subsection) changing the orientation of a component labeled by λ does not change the invariant.

The integral λ enjoys a particularly distinctive property: It generates H^\dagger as a free H -module. More specifically, if we define for each $h \in H$ the functional $f_h \in H^\dagger$ by

$$f_h(a) = \lambda(ah)$$

for all $a \in H$, then the map $h \mapsto f_h$ is one-to-one and onto. From this it follows that $f_\Lambda = \epsilon$, where Λ is an appropriately normalized left (and therefore right) integral for H .

Now consider, for H quasitriangular with $R = \sum_i a_i \otimes b_i$, a map from H^\dagger to H given by

$$D : f \mapsto \sum_{i,j} f(a_i b_j) b_i a_j.$$

Notice by Equation (5) this is a homomorphism. We say H is *factorizable* if this homomorphism D (the Drinfel'd map) is bijective. In that case the image of the integral λ must be a nonzero integral for H . Since $\lambda(D(\lambda)) = f_{D(\lambda)}(1) \neq 0$, we can normalize λ once and for all by the condition $\lambda(D(\lambda)) = 1$.

Theorem 4. *If H is a unimodular, factorizable quasitriangular Hopf algebra the invariant $K(L)$ of a link L all of whose components are labeled by the quantum character λ is an invariant of the compatibly framed three-manifold described by surgery on L .*

Proof. We have already argued that this invariant is unchanged by orientation reversal of any component. Invariance under spin Kirby move II follows from the normalization of λ and Property 1 of the list at the end of the previous subsection (the invariant of the Hopf link is $\lambda((D\lambda))$).

Before proving invariance under move I', consider the fragment in Figure 10. By repeated application of Equation (7) we see that the sum over all states of the values assigned to the fragment is $\sum_k a_k \otimes \Delta^{n-1}(b_k)$, where the n entries of the tensor product label the components from left to right at the bottom of the fragment. Similarly, using Equations (6) and (7), we see that the same fragment with the opposite parity is associated to $\sum_i S(b_i) \otimes \Delta^{n-1}(a_i)$, the mirror image of this fragment is associated to $\sum_i \Delta^{n-1}(S(b_i)) \otimes a_i$, and the mirror image with opposite parity by $\sum_i \Delta^{n-1}(a_i) \otimes b_i$.

With this in hand we see that the fragment on the left-hand side of Figure 5 corresponds (with the same convention of decorating the strands from left to right at the bottom of the picture) to

$$\begin{aligned} \sum_{i,j,k} a_i \otimes \lambda(a_j b_k) \Delta^{n-1}(b_k a_j b_i) &= \sum_i a_i \otimes \Delta^{n-1}(D(\lambda) b_i) \\ &= \sum_i a_i \otimes \epsilon(b_i) \Delta^{n-1}(D(\lambda)) = 1 \otimes \Delta^{n-1}(D(\lambda)) \end{aligned}$$

the last line by Equation (9). Of course, replacing each a_i in the above computation by $S(b_i)$ and every b_i with a_i , we see that the right-hand side equals the same thing, and thus the two fragments are interchangeable in the calculation of the invariant. \square

The most important special case of this construction arises when we consider the Drinfel'd double H_A of a finite-dimensional Hopf algebra A . It is well-known [31] that H_A is quasitriangular, unimodular and

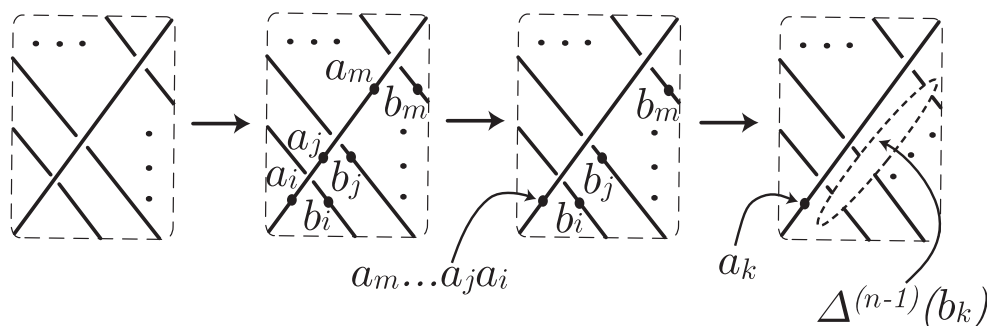


FIGURE 10. Hennings invariant of a cabled crossing

factorizable. Thus to every finite-dimensional Hopf algebra A this construction associates an invariant of compatibly framed (or spin) three-manifolds. There is ample empirical evidence, but as yet no proof, that this is exactly the invariant associated by Kuperberg to the Hopf algebra A in [25].

REFERENCES

- [1] C. Adams. *The Knot Book*. W. H. Freeman and Company, New York, 1994.
- [2] M. F. Atiyah. On framings of 3-manifolds. *Topology*, 29(1):1–7, 1990.
- [3] J. Barrett and B. Westbury. The equality of 3-manifold invariants. *Math. Proc. Cambridge Philos. Soc.*, 118:503–510, 1995.
- [4] J. Barrett and B. Westbury. Invariants of piecewise-linear 3-manifolds. *Transactions A. M. S.*, 348(10):3997–4022, 1996.
- [5] C. Blanchet. Invariants on three-manifolds with spin structure. *Comment. Math. Helv.*, 67(3):406–427, 1992.
- [6] C. Blanchet and G. Masbaum. Topological quantum field theories for surfaces with spin structure. *Duke Math. J.*, 82(2):229–267, 1996.
- [7] A. Bruguières. Catégories prémodulaires, modularisations et invariants des variétés de dimension 3. preprint, 1999.
- [8] G. Burde and H. Zieschang. *Knots*. Walter de Gruyter, New York, 1986.
- [9] S. Chung, M. Fukuma, and A. Shapere. Structure of topological lattice field theories in three dimensions. *Internat. J. Modern Phys. A*, 9(8):1305–1360, 1994.
- [10] R. Dijkgraaf and E. Witten. Topological gauge theories and group cohomology. *Commun. Math. Phys.*, 129:393–429, 1990.
- [11] V. G. Drinfel’d. On almost cocommutative Hopf algebras. *Lenigrad Math. J.*, 1(2):321–342, 1990.
- [12] R. Fenn and C. Rourke. On Kirby’s calculus of framed links. *Topology*, 18(1):1–15, 1979.
- [13] M. A. Hennings. Invariants of links and 3-manifolds obtained from Hopf algebras. *J. London Math. Soc.*, 2(54):594–624, 1996.
- [14] C. Kassel. *Quantum Groups*, volume 155 of *GTM*. Springer-Verlag, New York, 1995.

- [15] L. H. Kauffman. Hopf algebras and invariants of 3-manifolds. *J. Pure Appl. Alg.*, 100(1-3):73–92, 1995.
- [16] L. H. Kauffman and D. E. Radford. Invariants of 3-manifolds derived from finite dimensional Hopf algebras. *J. Knot Theory Ramifications*, 4(1):131–162, 1995.
- [17] T. Kerler. Mapping class group actions on quantum doubles. *Comm. Math. Phys.*, 168(2):353–388, 1995.
- [18] T. Kerler. Genealogy of non-perturbative quantum-invariants of 3-manifolds: the surgical family. In *Geometry and Physics (Aarhus, 1995)*, pages 503–547. Dekker, New York, 1997.
- [19] T. Kerler. Bridged links and tangle presentations of cobordism categories. *Adv. Math.*, 141(2):207–281, 1999.
- [20] R. Kirby. A calculus for framed links in S^3 . *Invent. Math.*, 45(1):35–56, 1978.
- [21] R. Kirby and P. Melvin. The 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2, \mathbb{C})$. *Invent. Math.*, 105(3):473–545, 1991.
- [22] R. C. Kirby. *The Topology of 4-Manifolds*, volume 1374 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.
- [23] A. A. Kirillov. On an inner product in modular tensor categories. *J. Amer. Math. Soc.*, 9(4):1135–1168, 1996.
- [24] G. Kuperberg. Involutionary Hopf algebras and 3-manifold invariants. *Internat. J. Math.*, 2(1):41–66, 1991.
- [25] G. Kuperberg. Noninvolutionary Hopf algebras and 3-manifold invariants. *Duke Math. J.*, 84(1):83–129, 1996.
- [26] V. Lyubashenko. Invariants of 3-manifolds and projective representations of mapping class groups via quantum groups at roots of unity. *Comm. Math. Phys.*, 172(3):467–516, 1995.
- [27] V. Lyubashenko. Modular transformations for tensor categories. *J. Pure Appl. Algebra*, 98(3):279–327, 1995.
- [28] M. Müger. Galois theory for braided tensor categories and the modular closure. preprint, 1999.
- [29] H. Murakami, T. Ohtsuki, and M. Okada. Invariants of three-manifolds derived from linking matrices of framed links. *Osaka J. Math.*, 29:545–572, 1992.
- [30] T. Ohtsuki. Invariants of 3-manifolds derived from universal invariants of framed links. preprint, 1993.
- [31] D. Radford. The trace function and Hopf algebras. *Journal of Algebra*, 163:583–622, 1994.
- [32] N. Yu Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127:1–26, 1990.
- [33] N. Yu Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103:547–597, 1991.
- [34] S. Sawin. Closed subsets of the Weyl alcove and TQFTs. preprint.
- [35] S. Sawin. Three-dimensional 2-framed TQFTs and surgery. preprint.
- [36] S. Sawin. Links, quantum groups, and TQFTs. *Bull. Amer. Math. Soc.*, 33(4):413–445, 1996.
- [37] S. Sawin. Jones-Witten invariants for nonsimply-connected Lie groups and the geometry of the Weyl alcove. To Appear in *Adv. Math.*, 2000.
- [38] B. Trace. On the Reidemeister moves of a classical knot. *Proc. Amer. Math. Soc.*, 89:722–724, 1983.

- [39] V. G. Turaev. *Quantum Invariants of Knots and 3-Manifolds*. Studies in Mathematics. Walter de Gruyter and Co., Berlin, 1994. ISBN 3-11-013704-6.
- [40] K. Walker. On Witten's 3-manifold invariants. (preprint).

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