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## SURFACE GROUPS WITHIN BAUMSLAG DOUBLES

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*Dedicated to Colin Maclachlan on the occasion of his 70th birthday*

*Abstract* A conjecture of Gromov states that a one-ended word-hyperbolic group must contain a subgroup that is isomorphic to the fundamental group of a closed hyperbolic surface. Recent papers by Gordon and Wilton and by Kim and Wilton give sufficient conditions for hyperbolic surface groups to be embedded in a hyperbolic Baumslag double  $G$ . Using Nielsen cancellation methods based on techniques from previous work by the second author, we prove that a hyperbolic orientable surface group of genus 2 is embedded in a hyperbolic Baumslag double if and only if the amalgamated word  $W$  is a commutator: that is,  $W = [U, V]$  for some elements  $U, V \in F$ . Furthermore, a hyperbolic Baumslag double  $G$  contains a non-orientable surface group of genus 4 if and only if  $W = X^2Y^2$  for some  $X, Y \in F$ .  $G$  can contain no non-orientable surface group of smaller genus.

*Keywords:* hyperbolic group; orientable surface group; quadratic word

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Secondary 20F65; 20E06; 20E07

### 1. Introduction

A *Baumslag double* is an amalgamated product of the form

$$G = F \star_{\{W=\bar{W}\}} \bar{F},$$

where  $F$  is a finitely generated free group,  $\bar{F}$  is an isomorphic copy,  $W$  is a non-trivial word in  $F$  and  $\bar{W}$  is its copy in  $\bar{F}$ . An orientable surface group of genus 2 is a Baumslag double and, in fact, Baumslag doubles were introduced in [2] to prove that surface groups are residually free. If  $G$  is a Baumslag double and if the identified word  $W$  is not a proper power in  $F$ , it follows from the combination theorems of Juhász and Rosenberger [7], Kharlampovich and Myasnikov [8] and Bestvina and Feighn [3] that the group  $G$  is hyperbolic. In fact, the Baumslag double  $G$  is hyperbolic if and only if  $W$  is not a proper power in  $F$  because  $W$  is a proper power in  $F$  if and only if  $\bar{W}$  is a proper power in  $\bar{F}$ .

An open conjecture of Gromov [6] states that a one-ended word-hyperbolic group must contain a subgroup that is isomorphic to the fundamental group of a closed hyperbolic

surface. Recent work by Gordon and Wilton [6] and by Kim and Wilton [9] gives sufficient conditions for hyperbolic surface groups to be embedded in a Baumslag double  $G$ . The work of Gordon and Wilton uses group cohomology and 3-manifold theory, while that of Kim and Wilton proceeds by realizing a Baumslag double as the fundamental group of a non-positively curved square complex.

In this paper, we use Nielsen cancellation methods based on the techniques in [12] to prove that a hyperbolic orientable surface group of genus 2 is embedded in a hyperbolic Baumslag double if and only if the amalgamated word  $W$  is a commutator: that is,  $W = [U, V]$  for some elements  $U, V \in F$ . Since an orientable surface group of genus 2 contains surface groups of all finite genus, it follows that  $G$  contains hyperbolic surface groups of all finite genus if and only if  $W$  is a commutator in  $F$ . Furthermore, a Baumslag double  $G$  contains a non-orientable surface group of genus 4 if and only if  $W = X^2Y^2$  for some  $X, Y \in F$ .

## 2. Main result

As mentioned in §1 it follows from the combination theorems of Juhasz and Rosenberger [7], Kharlampovich and Myasnikov [8] and Bestvina and Feighn [3] that the Baumslag double

$$G = F \star_{\{W=\bar{W}\}} \bar{F}$$

is hyperbolic if and only if the identified word  $W$  is not a proper power in  $F$ . We call such a group a *hyperbolic Baumslag double*. Here we assume that  $W$  is a reduced word in the free group  $F$ . Our main result is the following.

**Theorem 2.1.** *Let*

$$G = F \star_{\{W=\bar{W}\}} \bar{F}$$

*be a hyperbolic Baumslag double. Then  $G$  contains a hyperbolic orientable surface group of genus 2 if and only if  $W$  is a commutator: that is,  $W = [U, V]$  for some elements  $U, V \in F$ . Furthermore, a Baumslag double  $G$  contains a non-orientable surface group of genus 4 if and only if  $W = X^2Y^2$  for some  $X, Y \in F$ .*

Since an orientable surface group of genus 2 contains an orientable surface group of any finite genus as a subgroup, we immediately get the following corollary.

**Corollary 2.2.** *Let*

$$G = F \star_{\{W=\bar{W}\}} \bar{F}$$

*be a hyperbolic Baumslag double. Then  $G$  contains orientable surface groups of all finite genus if and only if  $W$  is a commutator.*

Before giving the proof we recall some material about surface groups and cyclically pinched one-relator groups.

A *surface group* is the fundamental group of a compact orientable or non-orientable surface. If the genus of the surface is  $g$ , then we say that the corresponding surface group also has genus  $g$ .

An orientable surface group  $S_g$  of genus  $g \geq 1$  has a one-relator presentation of the form

$$S_g = \langle a_1, b_1, \dots, a_g, b_g; [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle,$$

while a non-orientable surface group  $T_g$  of genus  $g \geq 1$  also has a one-relator presentation, which now has the form

$$T_g = \langle a_1, a_2, \dots, a_g; a_1^2 a_2^2 \cdots a_g^2 = 1 \rangle.$$

Much of combinatorial group theory originally arose out of the theory of one-relator groups and the concepts and ideas surrounding the Freiheitssatz or Independence Theorem of Magnus (see [11] or [10]). Going backwards, the ideas of the Freiheitssatz were motivated by the topological properties of surface groups [1].

The algebraic generalization of the one-relator presentation type of a surface group presentation leads to *cyclically pinched one-relator groups*. These groups have the same general form as a surface group and have proved to be quite amenable to study. In particular, a *cyclically pinched one-relator group* is a one-relator group of the following form:

$$G = \langle a_1, \dots, a_p, a_{p+1}, \dots, a_n; U = V \rangle,$$

where  $1 \neq U = U(a_1, \dots, a_p)$  is a cyclically reduced, non-primitive (i.e. not part of a free basis) word in the free group  $F_1$  on  $a_1, \dots, a_p$  and where  $1 \neq V = V(a_{p+1}, \dots, a_n)$  is a cyclically reduced, non-primitive word in the free group  $F_2$  on  $a_{p+1}, \dots, a_n$ .

Clearly, such a group is the free product of the free groups on  $a_1, \dots, a_p$  and  $a_{p+1}, \dots, a_n$ , respectively, amalgamated over the cyclic subgroups generated by  $U$  and  $V$ . Cyclically pinched one-relator groups have been shown to be extremely similar to surface groups [1].

A cyclically pinched one-relator group is hyperbolic if either  $U$  or  $V$  is not a proper power in its respective free group factor [3, 7, 8].

In [2], Baumslag introduced a double of a free group, now called a *Baumslag double*, in order to prove that orientable surface groups are residually free. In that paper he also proved that if neither  $U$  nor  $V$  is a proper power, then a cyclically pinched one-relator group is 2-free: that is, any 2-generator subgroup must be free. This was generalized by Rosenberger, who proved the following result [12, Theorem 3.3, p. 335] using Nielsen cancellation methods. This result is one of the bases for the proof of Theorem 2.1.

**Theorem 2.3 (Rosenberger [12]).** *Let  $G$  be a cyclically pinched one-relator group of the form*

$$G = \langle a_1, \dots, a_p, a_{p+1}, \dots, a_n; W = V \rangle,$$

where  $1 \neq W = W(a_1, \dots, a_p)$  is a cyclically reduced, non-primitive (i.e. not part of a free basis) word in the free group  $F_1$  on  $a_1, \dots, a_p$  and where  $1 \neq V = V(a_{p+1}, \dots, a_n)$  is a cyclically reduced, non-primitive word in the free group  $F_2$  on  $a_{p+1}, \dots, a_n$ . Suppose that neither  $W$  nor  $V$  is a proper power in its respective free group factor. Then we have the following.

- (a) Every subgroup  $H \subset G$  of rank 3 is free of rank 3.
- (b) Let  $H \subset G$  be a subgroup of rank 4. One of the following two cases then occurs.
- (i)  $H$  is free of rank 4.
  - (ii) If  $\{x_1, x_2, x_3, x_4\}$  is a generating system of  $H$ , then there is a Nielsen transformation from  $\{x_1, x_2, x_3, x_4\}$  to a system  $\{y_1, \dots, y_n\}$  with  $y_1, y_2 \in zF_1z^{-1}$  and  $y_3, y_4 \in zF_2z^{-1}$  for some  $z \in G$ . Moreover, there is a one-relator presentation for  $H$  on the generating system  $\{x_1, x_2, x_3, x_4\}$ .

Before presenting the proof we need two other ideas concerning Nielsen cancellation in free products with amalgamation. A word  $w \in F$ , where  $F$  is a free group on  $x_1, \dots, x_n$ , is *regular* if there exists no automorphism  $\alpha : F \mapsto F$  such that  $\alpha(w) = w'$ , when written as a word in  $x_1, \dots, x_n$ , contains fewer of the generators than  $w$  itself does. An ordered set  $U = \{u_1, \dots, u_n\} \subset F$  is regular if there exists no Nielsen transformation from  $U$  to a system  $U' = \{u'_1, \dots, u'_n\}$  in which one of the elements equals 1. This type of regularity is extended to free products with amalgamation in the following way. Suppose that  $G$  is a free product with amalgamation with factors  $H_1$  and  $H_2$  such that  $G = H_1 \star_A H_2$ . An ordered set  $U = \{u_1, \dots, u_n\} \subset G$  is then regular if there exists no Nielsen transformation from  $U$  to a system  $U' = \{u'_1, \dots, u'_n\}$  in which one of the elements is conjugate to an element of  $A$ .

The other crucial result for the proof of Theorem 2.1 is the following technical theorem [5, Theorem 5.3]. Recall that if  $F$  is a free group on  $X = \{x_1, \dots, x_n\}$ , then a reduced word  $w = w(x_1, \dots, x_n)$  is a *quadratic word* if each  $x_i$ , which appears in  $w$  as  $x_i$  or  $x_i^{-1}$ , appears exactly twice. For example, the surface group word of genus 2,  $[x_1, x_2][x_3, x_4] = x_1^{-1}x_2^{-1}x_1x_2x_3^{-1}x_4^{-1}x_3x_4$ , is a quadratic word.

**Theorem 2.4 (Fine et al. [5]).** *Suppose that  $G = H_1 \star_A H_2$  with  $H_1 \neq A \neq H_2$  and  $A$  malnormal in both  $H_1$  and  $H_2$ . Let  $F$  be a free group of rank  $n$  with  $1 \leq n \leq 4$  and let  $1 \neq w = w(x_1, \dots, x_n)$  be a regular quadratic word on the ordered basis  $X = \{x_1, \dots, x_n\}$ . Furthermore, let  $\phi : F \mapsto G$  be a homomorphism such that  $U = \phi(X)$  is regular in  $G$  and  $\phi(w) = 1$ . Then the pair  $(w, U)$  is Nielsen equivalent to a pair  $(w', U') = (\alpha(w), \alpha^{-1}(U))$  with  $\alpha : F \mapsto F$  an automorphism such that*

- (1)  $w' = w_1w_2$ , where  $w_1, w_2$  are also quadratic in  $F$ ,
- (2) for  $i = 1, 2$  we have that  $\phi(\alpha^{-1}(w_i))$  is conjugate to an element of  $A$  and
- (3) for  $i = 1, 2$  there is a  $\nu_i \in \{1, 2\}$  and a  $g_i \in G$  with  $\phi(\alpha^{-1}(x_j)) \in g_iH_{\nu_i}g_i^{-1}$  for each  $x_j$  that occurs in  $w_i$ .

We now give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Suppose that

$$G = F \star_{\{W=\bar{W}\}} \bar{F}$$

is a hyperbolic Baumslag double, where  $F$  is a free group on  $X = \{x_1, \dots, x_n\}$ . Since we are assuming hyperbolicity, we have that  $W$  and hence  $\bar{W}$  are not proper powers. Furthermore, if  $W$  were either trivial or primitive in  $F$ , then  $G$  would be a free group, so  $G$  could not contain a surface group that is either orientable or non-orientable. Furthermore, if  $G$  contains a surface group, then  $G$  cannot be a free group and hence  $W$  is neither trivial nor primitive. Therefore, we may assume that the amalgamated word  $W$  is neither trivial nor primitive.

We consider the orientable case first. Suppose that  $W = [u, v]$  in  $F$ . Then

$$W(\bar{W})^{-1} = [u, v]([\bar{u}, \bar{v}])^{-1} = [u, v][\bar{v}, \bar{u}] = 1.$$

Consider the subgroup  $H = \langle u, v, \bar{u}, \bar{v} \rangle$  of  $G$ . We can see that  $H$  cannot be a free group by applying Theorems 2.3 and 2.4 to the equation  $[u, v][\bar{u}, \bar{v}] = 1$  in  $G$  and from the fact that  $G$  does not contain a free abelian group of rank 2. Hence  $H$  is a one-relator group of rank 4 by Theorem 2.3. We show that a defining relation is precisely  $[u, v][\bar{v}, \bar{u}] = 1$ .

Let  $G$  be a cyclically pinched one-relator group of the form

$$G = \langle a_1, \dots, a_p, a_{p+1}, \dots, a_n; W = V \rangle$$

as in Theorem 2.3. Let  $H = \langle x_1, x_2, x_3, x_4 \rangle$  be a rank 4 subgroup of  $G$ . Within the proof of Theorem 2.3 [12, Theorem 3.3, pp. 335–340] it is shown that if  $H$  is not free, then not only is  $H$  a one-relator group but a method is described showing how to obtain a defining relation for  $H$  [12, p. 340]. This is done in the following manner. If there is a Nielsen transformation from  $\{x_1, x_2, x_3, x_4\}$  to a system where one element is conjugate to an element in the amalgamated subgroup, then  $H$  is free of rank 4. Now assume that  $H$  is not free of rank 4. Then, by the statement of Theorem 2.3,  $G$  is a one-relator group, and we may assume, possibly after a Nielsen transformation and a conjugation, that  $x_1, x_2$  are in  $F_1$ , the free group on  $a_1, \dots, a_p$ , that  $x_3, x_4$  are in  $F_2$ , the free group on  $a_{p+1}, \dots, a_n$ , and that  $W$  is in  $\langle x_1, x_2 \rangle$  or  $V$  is in  $\langle x_3, x_4 \rangle$ . Let  $W$  be in  $\langle x_1, x_2 \rangle$ . We consider the subgroup  $K = \langle V, x_3, x_4 \rangle$  in  $F_2$ . (Recall that  $W = V$  in  $G$ .)  $K$  cannot be a free group of rank 3 because otherwise  $H$  is free of rank 4. Hence  $K$  is a one-relator group in  $V, x_3, x_4$  and therefore  $H$  is a one-relator group in  $x_1, x_2, x_3, x_4$ . The relation is obtained as follows. Take the relation for  $K$  and replace  $V$  by  $W$  as a word in  $x_1, x_2$ .

If we apply this to the hyperbolic Baumslag double with  $W = [u, v]$ , we must consider the situation where we have a free group  $F = \langle a, b; \rangle$  of rank 2 generated by a system  $\{r, s, [a, b]\}$ . However, if  $F = \langle a, b; \rangle$  is generated by a system  $\{r, s, [a, b]\}$ , it follows from [12, Lemma 3.17] (see also [13, Hilfsatz 5]) that there is a free Nielsen transformation  $\mathcal{T}$  from  $\{r, s, [a, b]\}$  to  $\{a, b, [a, b]\}$ , where  $[a, b]$  is not replaced. Not replaced means that in all the elementary Nielsen transformations of which  $\mathcal{T}$  is composed, the commutator  $[a, b]$  either remains unchanged, is changed to  $([a, b])^{-1}$  or is put in a different location in the respective triple (see [12, pp. 335–340] for more details). In the hyperbolic Baumslag double the transformations are identical in the other factor. Therefore,  $[u, v] = [\bar{u}, \bar{v}]$  must be a defining relation for  $H$ . It follows that  $H$  is an orientable surface group of genus 2 and  $G$  contains such a subgroup.

Conversely, let  $H$  be a subgroup of  $G$  that is an orientable surface group of genus 2. Hence  $H$  has a presentation

$$H = \langle x_1, x_2, x_3, x_4; [x_1, x_2][x_3, x_4] = 1 \rangle.$$

Consider the system  $\{x_1, x_2, x_3, x_4\} \subset G$  and apply Nielsen cancellations within the amalgamated free product  $G$  with respect to the quadratic word  $v = [x_1, x_2][x_3, x_4]$ .

The system  $\{x_1, x_2, x_3, x_4\}$  is regular. That is, there is no Nielsen transformation from  $\{x_1, x_2, x_3, x_4\}$  to a system that contains an element that is conjugate in  $G$  to a power of  $W$  or  $\bar{W}$ . If the system  $\{x_1, \dots, x_n\}$  was not regular, then  $H$  would have to be a free group from [12, Lemma 3.1]. Now we apply Theorem 2.4 to  $X = \{x_1, x_2, x_3, x_4\}$  and  $w = [x_1, x_2][x_3, x_4]$ . Then  $w' = \alpha(w) = w_1 w_2$ , with  $w_1$  and  $w_2$  both quadratic words as described in Theorem 2.4. Since  $w$  is a product of commutators in the hyperbolic Baumslag double  $G$  and  $\alpha$  is an automorphism, it follows that  $w'$  is alternating in the same way as  $w$ : that is, each  $x_i$  occurs in  $w'$  exactly once as  $x_i$  and exactly once as  $x_i^{-1}$ . Since both  $w_1$  and  $w_2$  are quadratic and  $F$  is a non-abelian free group, this implies, up to conjugation and renaming, that  $w_1 = [x, y]$  for some  $x, y \in F$ . Recall that a free group word  $[a, b][c, d]$  is not Nielsen equivalent to a word  $r^2 s^2 t^2 p^2$ , otherwise an orientable surface group of genus 2 would be isomorphic to a non-orientable surface group of genus 4. That this cannot happen is clear from abelianization. Since the amalgamated subgroup  $A = \langle W \rangle$  is cyclic and  $w_1$  is conjugate to an element of  $A$ , it follows that  $[x, y]$  is conjugate to  $W^n$  for some non-zero  $n \in \mathbb{Z}$ . However, since a commutator in a free group is never a proper power (see [10, p. 52] or [4]) this implies that  $[x, y]$  is conjugate to  $W$  or  $W^{-1}$ . Since a conjugate of a commutator is also a commutator, it follows that  $W = [U, V]$  for some elements  $U, V \in F$ , proving the theorem in the orientable case.

The proof for the non-orientable case is almost identical except that when we get  $w = w_1 w_2$  we must have  $w_1 = x^2 y^2$  for some  $x, y \in F$ . We must also use the analogous argument that if a free group  $F = \langle a, b; \rangle$  is generated by a system  $\{r, s, a^2 b^2\}$ , then there is a free Nielsen transformation from  $\{r, s, a^2 b^2\}$  to  $\{a, b, a^2 b^2\}$  where  $a^2 b^2$  is not replaced. As in the orientable case, this follows from Lemma 3.17 in [12] and the remark immediately after that lemma.  $\square$

We note that a hyperbolic Baumslag double can never contain an orientable surface group of genus 1 (that is, a free abelian group of rank 2) and can never contain a non-orientable surface group of genus less than or equal to 3 by Theorem 2.4.

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