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Jones-Witten invariants for nonsimply connected Lie groups and the geometry of the Weyl alcove

Stephen F. Sawin
Fairfield University, ssawin@fairfield.edu

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JONES-WITTEN INVARIANTS FOR NONSIMPLY-CONNECTED LIE GROUPS AND THE GEOMETRY OF THE WEYL ALCOVE

STEPHEN F. SAWIN

ABSTRACT. The quotient process of Müger and Bruguières is used to construct modular categories and TQFTs out of closed subsets of the Weyl alcove of a simple Lie algebra. In particular it is determined at which levels closed subsets associated to nonsimply-connected groups lead to TQFTs. Many of these TQFTs are shown to decompose into a tensor product of TQFTs coming from smaller subsets. The "prime" subsets among these are classified, and apart from some giving TQFTs depending on homology as described by Murakami, Ohtsuki and Okada, they are shown to be in one-to-one correspondence with the TQFTs predicted by Dijkgraaf and Witten to be associated to Chern-Simons theory with a nonsimply-connected Lie group. Thus in particular a rigorous construction of the Dijkgraaf-Witten TQFTs is given. As a byproduct, a purely quantum groups proof of the modularity of the full Weyl alcove for arbitrary quantum groups at arbitrary levels is given.

INTRODUCTION

Since Witten's seminal paper [28] relating the Jones polynomial [11] to Chern-Simons field theory, the link and three-manifold invariants descendent from the Jones polynomial have admitted two apparently incompatible interpretations. On the one hand all can be defined combinatorially in terms of quantum groups [15, 21, 22], an algebraic language for rigorously and coherently computing them and proving their basic properties. Unfortunately, in this framework it is very difficult to relate the invariants to classical topology and geometry and, partly as a consequence, the invariants have answered very few questions which might have been asked before their appearance, the ultimate test of the significance of a new field. On the other hand they can be described geometrically as an ill-defined average over the space of connections [28]. This definition offers a beautiful and compelling intrinsically three-dimensional framework for the invariants which connects them to much of the exciting geometry and physics that has arisen over the past few decades. But this definition is completely nonrigorous because of its

reliance on the path integral, a heuristic technique of physics whose precise mathematical formulation is widely believed to be a problem we will leave to our grandchildren.

Perhaps the central problem of the subject is to unite these two viewpoints. A complete resolution of this problem would amount to a rigorous interpretation of the path integral in this particular case, and while this is arguably easier than such an interpretation in more general or more physically interesting situations, it should be viewed as a very long term goal. Still, much interesting progress has been made towards the goal of putting various aspects of the path integral formulation on a firmer mathematical footing [2, 3, 4, 16, 23].

Another strategy is to use the physics as a source of conjectures and of geometric objects we should expect to see revealed in the combinatorial structure if we look hard enough. This is the strategy of the current paper.

The geometry and the algebra part company in the first step of the construction of the invariants, in which the geometric construction begins with a compact semisimple Lie group, while the algebraic construction begins with a semisimple Lie algebra. These are almost, but not quite, in one-to-one correspondence. There are typically several Lie groups with the same Lie algebra, which differ only in their fundamental group. The invariants constructed from the Lie algebra correspond to the geometric construction with the simply-connected Lie group. Dijkgraaf and Witten [6] address the issue of the existence of the geometrically defined invariant for nonsimply-connected groups.

Let G be a connected, simply-connected compact simple Lie group with Lie algebra $\mathfrak{g}$, let Z be a subgroup of its center $Z(G)$, and let $G_Z = G/Z$ be the quotient. Recall $Z(G) = \mathbb{Z}_{l+1}$ if G is of type A_l; $Z(G) = \mathbb{Z}_2$ if G is of type B_l, C_l, or E_7; $Z(G) = \mathbb{Z}_3$ if G is of type E_6; $Z(G) = \mathbb{Z}_4$ if G is of type D_{2n+1}; $Z(G) = 1$ if G is of type E_8, F_4, or G_2; and $Z(G) = \mathbb{Z}_2 \times \mathbb{Z}_2$ if G is of type D_{2n}. G_Z like G is compact, simple and connected with Lie algebra $\mathfrak{g}$, but its fundamental group is isomorphic to Z. All connected compact simple Lie groups with Lie algebra $\mathfrak{g}$ arise in this fashion.

Dijkgraaf and Witten argue that to construct a Chern-Simons theory based on the group G_Z requires only a choice of an element of $H^4(BG_Z, \mathbb{Z})$, that is of the fourth cohomology of the classifying space of G_Z with integer coefficients. Of course the projection map from G to G_Z induces a homomorphism from $H^4(BG_Z, \mathbb{Z})$ to $H^4(BG, \mathbb{Z})$. In every case but one (see below) these cohomology groups are isomorphic to the integers, and thus the above homomorphism can be viewed as multiplication by an integer N. Dijkgraaf and Witten show that N is the

least N such that $N(\lambda, \lambda)/2$ is an integer for each fundamental weight λ corresponding to an element of Z, where $(\,\cdot\,,\,\cdot\,)$ is the inner product in the weight space. We will find it most convenient to index everything by the *level* $k \in H^4(BG, \mathbb{Z})$, viewed as an integer, and thus Dijkgraaf and Witten's work predicts a Chern-Simons theory associated to G_Z exactly when k is a multiple of N defined above.

The one exception to the above observation about $H^4(BG_Z, \mathbb{Z})$ is when G is the simply-connected group associated to the Dynkin diagram D_{2n} and Z is all of the center $\mathbb{Z}_2 \times \mathbb{Z}_2$. We will follow Dijkgraaf and Witten in not considering this case, although it should be extremely interesting (see, e.g., Felder, Gawedski and Kupianen [7]) and warrants further study.

The problem is then to construct the invariants from the quantum group perspective. Let us first briefly review how the Lie algebra appears in the construction due to Reshetikhin and Turaev [21, 22].

Beginning with a Lie algebra $\mathfrak{g}$, one deforms the universal enveloping algebra $U(\mathfrak{g})$ by a deformation depending on a complex parameter q to get an algebra $U_q(\mathfrak{g})$ (called a quantum group) which satisfies the axioms of a ribbon Hopf algebra. In practice these axioms mean its representation theory can be used to construct a system of link invariants.

For generic q the algebra $U_q(\mathfrak{g})$ is semisimple, but when q is a root of unity it becomes nonsemisimple and quite subtle. Unfortunately, the three-manifold invariants arise only at roots of unity: In fact the Witten invariants at level k discussed above correspond to $U_q(\mathfrak{g})$ when $q = \exp(2\pi i/(k + h))$, with h the dual Coxeter number of $\mathfrak{g}$.

Fortunately, the invariants do not depend on the quantum group directly, but only on a piece of its representation theory. In fact each of the representations we consider corresponds naturally to a representation of the original Lie algebra, and all the information we will need will be computed from the classical representation using classical data involving weights and root spaces. Any subset of this collection of representations satisfying certain properties (that it forms a *modular category*) can be used following the procedure of Reshetikhin and Turaev [22, 27] to construct a three-manifold invariant satisfying certain cut-and-paste axioms expected of topological quantum field theories (invariants satisfying these cut-and-paste axioms are called TQFTs in the literature).

A strategy thus naturally presents itself. Nonsimply-connected Lie groups can also be approached in terms of subsets of the set of representations of a Lie algebra. The finite-dimensional irreducible representations of $\mathfrak{g}$ can be indexed by a cone inside the weight lattice called

the Weyl chamber. The simply-connected group G associated to $\mathfrak{g}$ acts irreducibly on all these representations, and in particular each element of the center $z \in Z(G)$ acts on the representation indexed by a weight λ as the identity times the complex number $\chi_z(\lambda)$. In fact χ_z is a homomorphism from the weight lattice to the unit circle for each z, χ is a homomorphism from $Z(G)$ into the dual group of the weight lattice, and for each subgroup Z the group G_Z acts on exactly those representations whose weights lie in the sublattice annihilated by Z under χ. Thus our quantum surrogate for G_Z should be $U_q(\mathfrak{g})$ together with those representations which lie in the sublattice annihilated by Z. We have only to confirm that this set of representations gives a modular category exactly when k is a multiple of N as above.

This strategy has been pursued in the special case of the group $SO(3)$, which is $SU_2/(\mathbb{Z}_2)$, by Frohman and Kania-Bartoszyńskia [8] building on work of Kirby and Melvin [14]. Unfortunately, while in this case Dijkgraaf and Witten predict a TQFT and three-manifold invariant when k is a multiple of 4, and in fact suggest there should be some sort of spin TQFT and invariant for k an odd multiple of 2, what Frohman and Kania-Bartoszyńskia actually get is a modular category exactly when k is odd! We will see that this holds for a general Lie group: The appropriate set of representations forms a modular category only when k is relatively prime to Dijkgraaf and Witten's N. It is difficult to imagine a worse failure of the geometric predictions.

In fact there are two subtleties which bring the algebraic invariants into line with the geometric predictions, though there is some additional structure in the situation for quantum groups which is not readily apparent in the geometric point of view. The first subtlety is that in the cases where modularity is predicted but fails the failure is because of a trivial sort of redundancy in the category which can be readily quotiented out. The result of the quotient is a modular category, and hence a TQFT and three-manifold invariant.

The second subtlety is that many of the invariants coming from this quantum group construction can be factored as a product of invariants (in fact the factoring works at the level of TQFTs), one of which is a very simple invariant of the first homology studied by Murakami, Ohtsuki and Okada [20] and the other of which is the invariant associated to a smaller set of representations. In the end there is one prime invariant (in the sense that it admits no further decomposition as factors) for each TQFT conjectured by Dijkgraaf and Witten, and all other invariants that we construct are formed out of these. It is in this sense that the conjectures of Dijkgraaf and Witten are confirmed. Interestingly, in many cases the invariants of Dijkgraaf and Witten and the original

quantum group invariants constructed by Reshetikhin and Turaev are not the prime version, but the prime invariant times one of the homology invariants. Since the homology invariants of Murakami et al (and hence the invariants of Dijkgraaf and Witten) are often zero this means that on closed manifolds the prime invariants contain more topological information than those that seem to arise naturally in the geometric interpretation. On manifolds with boundary the prime theory can be recovered from the composite theory.

The first section of this paper reviews the work of Bruguières and Müger on constructing quotients of a ribbon category which are modular (this is quite distinct from the restriction to the Weyl alcove via the truncated tensor product of Reshetikhin and Turaev [22] and Andersen and Paradowski [1], which we take as our starting point). In general there is a certain subcategory whose objects, called degenerate, are obstructions to modularity, and which must form the kernel of the quotient. In the case at hand, where the ribbon categories possess a ∗-structure, the quotient is possible as long as all the degenerate objects are even, which is to say a change in the framing of components labeled by these objects does not change the invariant. For us, the degenerate objects will always be invertible under the tensor product, and in fact form a cyclic group, which makes the modular category, invariant and TQFT easy to describe concretely in terms of the original category. The description of the TQFT requires a much deeper immersion into the work of Müger and Bruguières than the rest of the paper, and invokes in detail the theory of TQFTs, and thus is far less self-contained than the rest of the article. For this reason, despite the centrality of the TQFT to the study of these invariants, the description is relegated to an appendix.

The second section deals with the quantum groups at roots of unity and their representation theory. It considers the Weyl alcove, the subset of weights corresponding to the representations of the quantum group that we are concerned with, and in particular the isometries of the Weyl alcove (the weight lattice sits naturally in a Euclidean space). Of central importance are the invertible elements, which form the orbit of the trivial weight under isometries of the Weyl alcove. We prove that all degenerate objects in these categories are invertible, and we identify these degenerate objects and determine when they are even. Most of the work of the section involves a careful understanding of the so-called truncated tensor product of representations, and relies on a crucial formula of Andersen and Paradowski, generalizing a classical formula for the ordinary tensor product of representations to the quantum case. Isometries and invertible elements of the Weyl alcove were

used for very similar ends in a paper by Felder et al. [7] to address Wess-Zumino-Witten theory for nonsimply-connected Lie groups. The section gives a complete analysis of when a TQFT can be constructed from the representations of any quantum group associated to any classical simple, connected, compact Lie group at any level. In particular, this section offers a prof purely in the language of quantum groups that the full set of representations forms a modular category and hence a TQFT. The proofs in the literature [15, 27] all rely essentially on a result of Kac and Peterson from the theory of affine Lie algebras [12]. Bruguières has also constructed modular categories associated to the group PGL_n using his results.

Section Three identifies under what circumstances the categories and TQFTs constructed factor into simpler theories, and identifies the factors. The results of this section were first suggested to the author by E. Witten in private conversation. The decomposition exhibited here was observed previously by Kirby and Melvin [14] on the level of invariants for the group $SU(2)$, where at odd levels the theory is a product of the prime theory (which they call the $SO(3)$ theory) and Murakami et al's invariant Z_2. Again the more technical work with TQFTs appears in the appendix.

1. Semisimple Ribbon *-Categories and Quotients

We will follow the notation of Kirillov [15] in this section, and quote the basic results from that paper, but other good sources for the theory of ribbon categories include books by Kassel and Turaev [13, 27]. This section relies heavily on the work of Müger [19]. Similar results were obtained independently by Bruguières [5], but we will follow Müger, whose language is more in line with the rest of this paper.

1.1. **Semisimple ribbon *-categories.** A *rigid monoidal category* is a category $\mathfrak{C}$ together with a bifunctor $\otimes\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ which is an associative multiplication with an identity object and morphism (we will assume our category has been 'strictified' as in MacLane [17]) and a notion of duals compatible with $\otimes$. The example on which to base one's intuition is the category of finite-dimensional vector spaces and

linear maps, where $\otimes$ is ordinary tensor product and the duals are vector space duals.

The category is called a semisimple $*$-category if the hom sets are vector spaces over $\mathbb{C}$, with composition and $\otimes$ acting as bilinear operations, if there are direct sums and kernels (i.e., idempotents in the hom space of an object always factor through a subobject), and there is an antilinear involution $*$ between $\hom(\lambda, \gamma)$ and $\hom(\gamma, \lambda)$ which is contravariant ($(f \circ g)^* = g^* \circ f^*$), monoidal ($(f \otimes g)^* = f^* \otimes g^*$), positive ($f^* \circ f = 0 \implies f = 0$), and consistent with the duality in the appropriate sense. Here the example to keep in mind is the category of representations of a C^*-algebra whose morphisms are linear maps between representations which commute with the algebra's action, and whose simple objects are the irreducible representations. The $\otimes$ and duality structure occur naturally if the C^*-algebra is a Hopf algebra.

Finally, a semisimple rigid monoidal $*$-category $\mathfrak{C}$ is a *semisimple ribbon $*$-category* if every pair of objects λ and γ admits a unitary isomorphism $R_{\lambda,\gamma} \colon \lambda \otimes \gamma \to \gamma \otimes \lambda$ and every object λ admits a unitary isomorphism $\theta_\lambda \colon \lambda \to \lambda$ satisfying certain relations found in Kassel [13]. In the presence of the $*$-structure the θ-morphism can be constructed from the R-morphisms. The relations are of course designed to guarantee that there is a functor whose range is $\mathfrak{C}$ and whose domain is the category of framed tangles with components labeled by objects of $\mathfrak{C}$ [15, 26], with a simple crossing corresponding to the R morphism and a full twist to the θ morphism (see Figure 1). Note that the 'quantum dimension' $\operatorname{qdim}(\lambda)$ of each simple object λ, which is defined in [15] and corresponds to the invariant of the zero-framed unknot labeled by V (See Figure 1), is always a positive real (in fact ≥ 1).

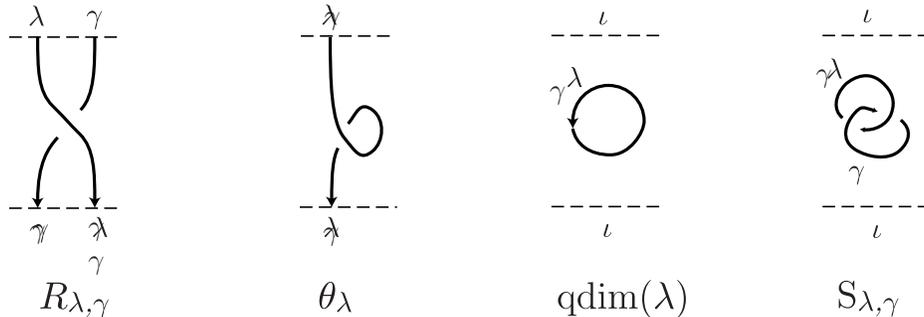$$R_{\lambda,\gamma} \qquad \theta_\lambda \qquad \operatorname{qdim}(\lambda) \qquad S_{\lambda,\gamma}$$

FIGURE 1. Certain important knots and tangles and their image under the link invariant

Consider a semisimple ribbon $*$-category $\mathfrak{C}$. Let Γ, the *label set* of $\mathfrak{C}$, be the set of all isomorphism classes of simple objects in $\mathfrak{C}$. We will

call the identity object for $\otimes \iota$, and will use the same name to refer
to its isomorphism class in in Γ. The dual gives an involution on Γ
which we call $\dagger$. Now for each $\gamma \in \Gamma$ the one-dimensional $\hom(\gamma, \gamma)$
can be canonically identified with $\mathbb{C}$ as a C^*- algebra, so the morphism
θ_γ corresponds to some complex number C_γ of modulus 1 such that
θ_γ is C_γ times the identity. Also if $\lambda, \gamma \in \Gamma$, then $\lambda \otimes \gamma$ is isomorphic
to a sum $\bigoplus_{\eta \in \Gamma} N_{\lambda,\gamma}^\eta \eta$, where the nonnegative integers $N_{\lambda,\gamma}^\eta$ represent
multiplicities. In this and the sequel, we freely confuse simple objects
with their isomorphism classes, trusting the sophistication of the reader
to unravel the subtleties.

Some facts about these numbers and their relation to the invariant
will be useful. We have $\mathrm{qdim}(\lambda^\dagger) = \mathrm{qdim}(\lambda)$, $\mathrm{qdim}(\lambda \otimes \gamma) = \mathrm{qdim}(\lambda) \cdot$
$\mathrm{qdim}(\gamma)$, $N_{\lambda,\gamma}^\eta = N_{\gamma,\lambda}^\eta = N_{\gamma,\eta^*}^{\lambda^\dagger} = N_{\lambda^\dagger,\gamma^\dagger}^{\eta^\dagger}$, $N_{\lambda,\gamma}^\iota = \delta_{\lambda,\gamma^\dagger}$, and $C_\lambda = C_{\lambda^\dagger}$,
$C_\iota = 1$.

A ribbon category yields an invariant of labeled framed ribbon graphs.
More specifically, consider an oriented graph embedded smoothly in S^3
with a well-defined normal bundle (that is, the edges incident to a ver-
tex are all tangent to a single plane at that point), equipped with a
nonzero section of the normal bundle and a choice of edge at each ver-
tex. Label each edge by an object of the category. Notice the framing
gives a cyclic ordering on the edges incident to a vertex, and starting
at the chosen edge makes this a total ordering. Thus to each vertex
we can associate the object $\overline{\lambda}_1 \otimes \overline{\lambda}_2 \otimes \cdots \otimes \overline{\lambda}_n$ where λ_i is the label of
the ith incident edge and $\overline{\lambda}_i$ is λ_i or $\lambda_i^\dagger$ according to whether the edge
is oriented towards or away from the vertex. Label each vertex by an
element of $\hom(\gamma, \iota)$, where γ is this associated object. The category
associates to this framed, labeled graph a number which is invariant
under ambient isotopy of the framed graph. If a connected component
of the graph contains only bivalent vertices and thus is homeomorphic
to a circle (we call such components *link components*) and the edge la-
bel is simple, we can ignore the vertices and view it as a circle labeled
by an object of the category (up to an overall scale factor). Thus we get
in particular a framed labeled link invariant. The following properties
of the invariant will be important to us:

1. the invariant of a graph with an edge labeled by $\lambda \oplus \gamma$ is the sum of
   the invariants of the same graph with that edge labeled by λ and
   γ respectively, the labels on the adjacent vertices being projected
   appropriately,
2. if the label of an edge is replaced by an isomorphic object and the
   labels of the adjacent vertices are composed with the isomorphism
   in the obvious way, the invariant is unchanged. In particular,

    link components can be unambiguously labeled by elements of Γ, rather than objects,

3. the invariant of a graph with an edge labeled by ι is the same as the invariant of the graph with that edge deleted,
4. the invariant of a graph with an edge labeled by λ is the invariant of the graph with the orientation of that edge reversed and the label replaced by $\lambda^\dagger$, the labels of the adjacent vertices remaining the same,
5. the invariant of a graph with a link component labeled by $\lambda\otimes\gamma$ is the invariant of the graph with that component replaced by two parallel components (according to the framing) labeled by λ and γ respectively,
6. The invariant of the connected sum of two graphs along edges labeled by a simple object λ is the product of the invariants of the two graphs divided by $\mathrm{qdim}(\lambda)$, where the connect sum consists of cutting the chosen edges and reconnecting them as in the definition of connected sum for knots, and
7. if a sphere intersects a ribbon graph only transversely at two edges labeled by simple objects and oriented oppositely, the two objects must be isomorphic or the invariant is zero.

We say a subset Γ' of Γ is *closed* if it is closed under the duality involution and if whenever $\lambda, \gamma \in \Gamma'$ and $N^\eta_{\lambda,\gamma} \neq 0$, we have $\eta \in \Gamma'$ (i.e., the product of elements of Γ' is a sum of elements of Γ').

Proposition 1. *If Γ' is a closed subset of Γ, the full subcategory of $\mathfrak{C}$ whose objects are sums of objects in the isomorphism classes in Γ' is again a semisimple ribbon category.*

Proof. Immediate from the definition. $\square$

1.2. **Degenerate objects.** Suppose $\mathfrak{C}$ is a semisimple ribbon $*$-category with label set Γ. For each $\lambda, \gamma \in \Gamma$ define $S_{\lambda,\gamma}$ to be the value of the invariant of the zero-framed Hopf link with components labeled by λ and γ respectively (see Figure 1). Thus

$$(1) \qquad S_{\lambda,\gamma} = \sum_\eta N^\eta_{\lambda,\gamma} \, \mathrm{qdim}(\eta) C_\eta C_\lambda^{-1} C_\gamma^{-1}.$$

By Properties 1-7 above $S_{\lambda,\gamma} = S_{\gamma,\lambda} = S_{\lambda^\dagger,\gamma^\dagger}$, $S_{\lambda,\iota} = \mathrm{qdim}(\lambda)$, $S_{\mu\otimes\lambda,\gamma} = \sum_\eta N^\eta_{\mu,\lambda} S_{\eta,\gamma}$ and $S_{\mu\otimes\lambda,\gamma} = S_{\mu,\gamma} S_{\lambda,\gamma} / \mathrm{qdim}(\gamma)$. The matrix of numbers $S_{\lambda,\gamma}$ is called the S-matrix. Recall that a *modular category* is a semisimple ribbon category with a finite label set Γ such that the S-matrix is invertible (see [15]).

    Müger studies objects λ such that $R_{\lambda,\gamma} = R_{\gamma,\lambda}^{-1}$ for all γ in the category, and calls such objects *degenerate*. He proves

Theorem 1. *A semisimple ribbon ∗-category is modular if and only if all its degenerate simple objects are isomorphic to the trivial object.*

Suppose one is computing the link invariant from a projection of a link with one component labeled by a degenerate object. By the definition, the link invariant is unchanged by switching any of the crossings in which that component participates from an overcrossing to an undercrossing or vice-versa. It is a standard observation of knot theory that in that case the component can be unlinked from the other components and unknotted except for the framing. Thus the invariant of the original link is the same as the invariant of the link with that component deleted times $\text{qdim}(\lambda)C_\lambda^m$, where m is the framing of that component. Moreover, from the fact that $R_{\lambda,\lambda}^2 = 1$ we conclude that $C_\lambda = \pm 1$. If $C_\lambda = 1$ we call λ *even* and we see that λ provides no link information, and as far as the link invariant is concerned behaves as if it were a multiple of the trivial object.

One might reasonably suspect that one can 'quotient out' by the even degenerate objects to get a smaller ribbon category that in some sense is a minimal ribbon category associated to the link invariant, and further, that by doing so one would make the resulting quotient modular. The first suspicion is correct, and the second is correct as long as all degenerate objects are even. Müger's proof of these results uses the fact that the degenerate objects form a symmetric ∗-subcategory and the result of Doplicher and Roberts that such a category is always the representation category of a compact group. Bruguières uses the related Tannaka-Krein theorem to prove a similar result which does not assume the ∗-structure but replaces it with assumptions that amount to saying degenerate objects can be represented as vector spaces. The universal characterization of the quotient below appears in Bruguières' work and not in Müger's, but it is an easy consequence of results in the latter.

Theorem 2. *If $\mathfrak{C}$ is a semisimple ribbon ∗-category such that all of its degenerate objects are even, then there exists a modular category $\mathfrak{C}'$ and a ribbon ∗-functor from $\mathfrak{C}$ to $\mathfrak{C}'$, with the property that every ribbon ∗-functor from $\mathfrak{C}$ to a modular category factors through this functor. In particular the link invariant of a link with components labeled by objects of $\mathfrak{C}$ is the same as that for the link labeled by their image objects in $\mathfrak{C}'$.*

1.3. **Invertible objects.** In general, the quotient category is quite complicated, and its relation to the original category murky. But in the special case when all of the degenerate objects are invertible, which

is the case for all ribbon categories arising from quantum groups, the relationship can be described much more explicitly.

An element $\lambda \in \Gamma$ is called *invertible* if $\lambda \otimes \lambda^\dagger = \iota$. The set of invertible objects form a group under tensor product, and if λ is invertible and γ is simple then $\lambda \otimes \gamma$ is also simple, because its product with λ^* is simple. Thus each invertible object λ corresponds to a map ϕ_λ on objects defined by $\phi_\lambda(\gamma) = \lambda \otimes \gamma$ which descends to a bijection on Γ. The map ϕ_λ satisfies the relation

$$(2) \qquad \phi_\lambda(\gamma \otimes \gamma') = \phi_\lambda(\gamma) \otimes \gamma'.$$

The set of maps ϕ_λ on Γ forms a group under composition isomorphic to the group of invertible elements. We should caution that our confusion of objects and isomorphism classes here generates a minor subtlety: Isomorphic invertible elements generate distinct maps of objects, but all these maps descend to the same map on Γ, which we associate to the isomorphism class of the original objects.

Suppose $\mathfrak{C}$ is a semisimple ribbon $*$-category all of whose degenerate objects are both even and invertible. The set of isomorphism classes of such objects forms an abelian group Z under $\otimes$. Suppose further that this group Z is cyclic, so that in particular its second group cohomology is trivial. Z acts on Γ by $\otimes$, and associated to each element of Γ is its orbit and the stabilizer subgroup of Z. Müger shows that in the functor of Theorem 2 all the simple objects of an orbit get sent to the same object in $\mathfrak{C}'$, and that this object is the direct sum of a set of simple objects in $\mathfrak{C}'$ in one-to-one correspondence to the stabilizer (in general there is a multiplicity depending on a 2-cocycle, but because of the cyclicity of the group, this is one).

In any ribbon category with finitely many isomorphism classes of simple objects we can compute the very important link invariant

$$I(L) = \sum_{\gamma_1, \dots, \gamma_n \in \Gamma} \prod_{i=1}^{n} \mathrm{qdim}(\gamma_i) F_{\gamma_1, \dots, \gamma_n}(L)$$

where L is a link with n components, the sum is over all ways of choosing n isomorphism classes of simple objects out of Γ, and $F_{\gamma_1, \dots, \gamma_n}(L)$ is the invariant of L with the n components labeled by $\gamma_1, \dots, \gamma_n$ respectively (or more properly, representative objects of each of those classes). If we extend the invariant to allow formal linear combinations of objects as labels (the invariant being linear on each label) this is the invariant of L with each component labeled by $\omega = \sum_{\gamma \in \Gamma} \mathrm{qdim}(\gamma)\gamma$.

The importance of I is that in the case of a modular category, if L is a surgery presentation of a 2-framed three-manifold M with n

components and a linking matrix with signature σ then

$$(3) \qquad\qquad I(L)/I(H)^{n/2}$$

depends only on M and not on L, where H is the Hopf link. Furthermore

$$(4) \qquad\qquad (I(N)/I(P))^{\sigma/2} I(L)/I(H)^{n/2}$$

depends only on the underlying three-manifold and not on the 2-framing, where P is the $+1$ framed unknot and N is the -1 framed unknot. This can be written in the more congenial form

$$\left(\frac{I(N)}{|I(N)|} \right)^{\sigma} I(L)/|I(N)|^n$$

using the fact that $I(H) = I(P)I(N)$ and in a $*$-category $I(P) = \overline{I(N)}$.

Proposition 2. *Let $\mathfrak{C}$ be a ribbon $*$-category such that all of its degenerate objects are even and invertible, and such that the associated abelian group Z is cyclic. Then $I(L) = |Z|^n I'(L)$, where I' is the corresponding invariant for the quotient category $\mathfrak{C}'$, and therefore Equations (3) and (4) give 2-framed and ordinary three-manifold invariants.*

Proof. Notice all simple objects in one orbit give the same link invariant, so we could as well compute I by taking the sum only over a representative γ of each orbit class, and replacing the factor $\mathrm{qdim}(\gamma)$ with $|Z|/|S_\gamma| \mathrm{qdim}(\gamma)$, where S_γ is the stabilizer of γ and thus $|Z|/|S_\gamma|$ is the number of elements in the orbit of γ. Because the functor is a ribbon functor we can replace the invariant associated with $\mathfrak{C}$ with the invariant associated with $\mathfrak{C}'$, if we replace the label γ on L with its image under the functor. This image is the direct sum of $|S_\gamma|$ many simple objects, each with quantum dimension $\mathrm{qdim}(\gamma)/|S_\gamma|$. Decomposing the invariant into a sum of invariants, each with the link labeled by a single one of these simple objects, we obtain $|Z|^n I'(L)$. $\qquad\square$

Remark 1. The result above should hold more generally. In fact it is easy to check that in any ribbon *-category all of whose degenerate objects are even, the quantities (3) and (4) are respectively 2-framed and ordinary three-manifold invariants. It is to be expected that they are equal to the corresponding quantities in the quotient, but the proof may be more difficult.

Of course we wish to describe the entire TQFT in terms of the original category, just as we have here described the three-manifold invariant. In principle this is straightforward, but in practice to do this concretely requires much more detail both about Müger's construction

and the construction of TQFTs from modular categories than we will use in the rest of the article. These details appear in Appendix A.1, where a precise description of the TQFT is given.

Remark 2. The term quotient which we use to denote the modularization is very appropriate to the language of link invariants and TQFTs, but is not entirely accurate on the level of the category. In fact Müger proves that the functor from $\mathfrak{C}$ to $\mathfrak{C}'$ is an equivalence from $\mathfrak{C}$ to $(\mathfrak{C}')^{\hat{Z}}$, the subcategory of $\mathfrak{C}'$ invariant under the action of the dual group to Z.

2. The Geometry of the Weyl Alcove

This section relies heavily on the work of Andersen and Paradowski [1] and of Kirillov [15], which is summarized here.

2.1. **Quantum groups and the Weyl alcove.** Let $\mathfrak{g}$ be a simple Lie algebra with Dynkin diagram different from D_{2n}, and let $U_q(\mathfrak{g})$ be its quantized universal enveloping algebra. This is defined exactly as in Kirillov with our q equal to the square of his q except we will normalize the inner product on the Lie algebra so as to give long roots length 2 (he normalizes so that *short* roots have length 2). This normalization has the properties that it agrees with Kirby-Melvin and Reshetikhin-Turaev [14, 22] for $U_q(\mathrm{su}_2)$ and that it makes the quantum group $U_q(\mathfrak{g})$ a modular Hopf algebra with the standard set of representations exactly when q is a root of unity.

We will need some notation from Lie algebra theory, most of which is taken from Humphreys [10], an excellent general reference on the subject. Let r be the rank of $\mathfrak{g}$ and let $\{\alpha_i\}_{i \leq r}$ be the simple roots of $\mathfrak{g}$. The weight lattice Λ contains the sublattice Λ_r spanned by the roots, and we will be especially concerned with subgroups of the fundamental group Λ/Λ_r, because each such subgroup corresponds to a sublattice containing Λ_r. The center $Z(G)$ of the simply-connected group G with Lie algebra $\mathfrak{g}$ imbeds via the map χ defined in the introduction into the dual group to Λ, and in fact since each element of the center acts trivially on the representations in the root lattice, χ descends to an isomorphism from $Z(G)$ to the dual group of the fundamental group Λ/Λ_r, which isomorphism we will also denote by χ.

The Weyl group is denoted by $\mathfrak{W}$, and the set of weights in the fundamental Weyl chamber is called Λ^+ (we will loosely refer to this set itself as the Weyl chamber). Half the sum of the positive roots is called ρ (Humphreys calls this δ), and the unique long root in the Weyl chamber is called θ. This root corresponds to the adjoint representation of $\mathfrak{g}$. The dual Coxeter number h is defined to be $(\rho, \theta) + 1$, the value of

the quadratic Casimir on the adjoint representation. The *fundamental weights* $\{\lambda_i\}_{i \leq r}$ are given by $(\lambda_i, \alpha_j) = \delta_{i,j}(\alpha_i, \alpha_i)/2$.

Let $q = e^{2\pi i/(k+h)}$, for some natural number k. Kirillov shows that the category of representations of the quantum group $U_q(\mathfrak{g})$ corresponding to the weights in the *Weyl alcove* Λ_0, i.e. those λ in the Weyl chamber such that $(\lambda, \theta) \leq k$, form a semisimple ribbon category if the ordinary tensor product is replaced by the *truncated tensor product,* $\otimes$, which returns the the ordinary tensor product quotiented by the maximal tilting submodule [1]. Considered as a multiplication on the additive group of isomorphism classes of representations spanned by those in the Weyl alcove (with direct sum as addition) it is commutative, associative, distributive and determined by

$$(5) \qquad V_\lambda \otimes V_\gamma \cong \bigoplus_{\eta \in \Lambda_0} N_{\lambda,\gamma}^\eta V_\eta,$$

where $N_{\lambda,\gamma}^\eta$ are nonnegative integers representing multiplicities. The principal result we use from Andersen and Paradowski [1] is their formula for these numbers, a variation on Racah's formula for tensor product of classical representations

$$(6) \qquad N_{\lambda,\gamma}^\eta = \sum_{\sigma \in \mathfrak{W}_0} (-1)^\sigma m_\gamma(\lambda - \sigma(\eta))$$

where $m_\lambda(\mu)$ is the dimension of the μ weight space inside the classical representation of highest weight λ, $\mathfrak{W}_0$ is the quantum Weyl group, which is generated by reflection about the hyperplanes $\{x|(x+\rho, \alpha_i) = 0\}$ for each simple root α_i together with $\{x|(x,\theta) = k+1\}$, and $(-1)^\sigma$ is plus or minus one, according to whether σ is an even or odd product of these simple reflections. Notice that the weight 0, representing the trivial representation, is the identity object for the truncated tensor product, and thus is what in the last section was referred to by ι. Also

$$(7) \qquad C_\lambda = q^{(\lambda, \lambda + 2\rho)/2}.$$

and $\mathrm{qdim}(\lambda) > 0$.

2.2. **The invertible elements of the Weyl alcove.** The geometric definition of the truncated tensor product in (6) gives the invertible elements a special geometric role. We will see in the next subsection that degenerate simple objects are always invertible, and thus that invertible elements will be of central importance in the sequel.

Theorem 3. *There is an injection ℓ from $Z(G)$ to the fundamental weights of the Weyl alcove such that z acts on the classical representation V_γ as $\exp(2\pi i(\gamma, \ell(z))) \cdot \mathrm{id}_\gamma$. The fundamental weights λ_i in the*

image of ℓ are exactly those for which $(\lambda_i, \theta) = 1$ and the associated root α_i is long, and are also exactly those for which there is a unique element τ_i of the classical Weyl group taking the standard base to the base $\{\alpha_j\}_{j \neq i} \cup \{-\theta\}$. If we define $\phi_i(\gamma) = k\lambda_i + \tau_i(\gamma)$, then ϕ_i is an isometry of the Weyl alcove and of the simplex $\{\lambda : (\lambda, \alpha_j) \geq 0 \text{ and } (\lambda, \theta) \leq k\}$ and $\phi_i(\lambda \otimes \gamma) = \phi_i(\lambda) \otimes \gamma$ (i.e., ϕ_i satisfies Equation (2)). If we use k also to represent the map on the weight lattice which multiplies each weight by the number k, then $k\ell$ is a homomorphism in the sense that $k\ell(zz') = k\ell(z) \otimes k\ell(z')$. Weights in the range of $k\ell$ can be characterized as extreme points of the simplex $\{\lambda : (\lambda, \alpha_j) \geq 0 \text{ and } (\lambda, \theta) \leq k\}$ such that a neighborhood of the weight 0 in the simplex is isometric to a neighborhood of the extreme point in the simplex, the isometry being given by ϕ_i.

Remark 3.

- This ℓ is the isomorphism referred to in the introduction. Specifically, Dijkgraaf and Witten predict Chern-Simons theories for G_Z when Z is a subgroup of $Z(G)$ and k is such that $k(\ell(z), \ell(z))/2$ is an integer for each $z \in Z$.
- The homomorphism property of $k\ell$ shows its range consists of invertible objects, and of course ϕ_i is the map satisfying Equation (2) which is associated to the invertible object $k\lambda_i$. It is almost the case that these are all the invertible elements of the Weyl alcove. In fact it is shown in [24] that the only case in which these are not all of the invertible elements is for the quantum group of type E_8 at level $k = 2$, when the fundamental weight λ_1 is invertible but not in the range of $k\ell$. In fact, in this case the associated map ϕ is an isometry of the Weyl alcove, but *not* of the simplex.
- The local isometry condition says more intuitively that the range of $k\ell$ is the set of 'sharp corners' of the simplex. from the proposition it follows that the group of isometries of the simplex is the semidirect product of the group of isometries of the Weyl chamber (which correspond naturally to automorphisms of the Coxeter graph) with the group of maps $\{\phi_i\}$.
- Of course more generally there is a bijection from the full set of fundamental weights to the full set of extreme points (or corners) of the simplex given by $\lambda_i \mapsto k\lambda_i/(\lambda_i, \theta)$. For appropriate k these corners are weights and will figure prominently in the next section.
- The maps ℓ, k, and $k\ell$, their ranges and domains and their relation to the geometry of the simplex are illustrated in the case of B_2 for some arbitrary k in Figure 2. The map ϕ_i associated to the

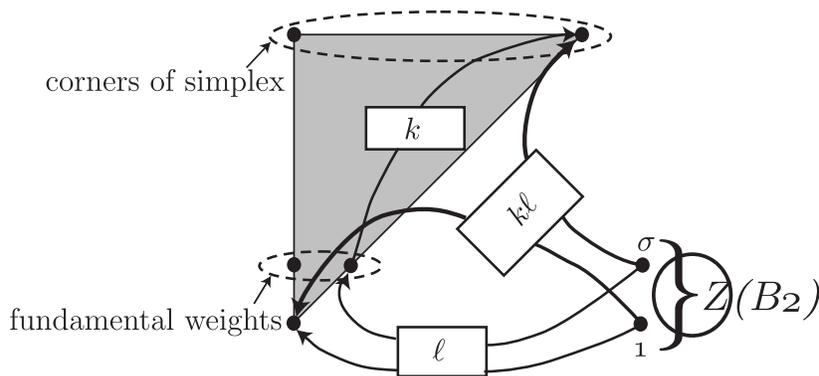nontrivial element σ of $Z(B_2) = \mathbb{Z}_2$ is of course the reflection about the diagonal.

FIGURE 2. The maps ℓ, k, and $k\ell$ for the case B_2

Lemma 1. *The set of fundamental weights λ_i such that α_i is long and $(\theta, \alpha_i) = 1$ is the same as the set such that there is a unique element τ_i of the classical Weyl group taking the standard base to the base $\{a_j\}_{j \neq i} \cup \{-\theta\}$. If e is the homomorphism from Λ to $(\Lambda/\Lambda_r)^*$, where $(\Lambda/\Lambda_r)^*$ is the dual group to the weight lattice modulo the root lattice, which sends each weight λ to the homomorphism e_λ given by $e_\lambda(\gamma) = \exp(2\pi i(\lambda, \gamma))$, then e is a bijection when restricted to this set.*

Proof. See the end of Section 2.2. □

Proof of Theorem 3. It is well known that $Z(G)$ is isomorphic to the group $(\Lambda/\Lambda_r)^*$ by the map sending z to the homomorphism χ_z on Λ such that z acts on the classical representation V_γ by $\chi_z(\gamma) \operatorname{id}_\gamma$ (this is clearly a homomorphism descending to $(\Lambda/\Lambda_r)^*$, and is injective by the faithfulness of the left regular representation. That the domain and range have the same dimension is shown in [10]). Thus by Lemma 1 we can construct a bijection ℓ from $Z(G)$ to the set of fundamental weights λ_i meeting the two characterizations of the proposition. This gives the first two sentences of the proposition. For the rest, we argue as follows. We first show that the map ϕ_i defined in the proposition satisfies the conditions of the proposition. We then use this to give the characterization of the range of $k\ell$ in the last sentence of the proposition. Finally this will allow us to show that $k\ell$ is a homomorphism.

The map ϕ_i is certainly an isometry of the weight space taking the weight 0 (i.e. the object ι) to $k\lambda_i$. The image of the elements of the

Weyl alcove are those weights γ such that $(\gamma - k\lambda_i, \alpha_j) \geq 0$ for $j \neq i$, $(\gamma - k\lambda_i, -\theta) \geq 0$, and $(\gamma - k\lambda_i, -\alpha_i) \leq k$. Since $(k\lambda_i, \alpha_j) = 0$ for $j \neq i$ and $(k\lambda_i, \theta) = (k\lambda_i, \alpha_i) = k$, such γ are exactly the elements of the Weyl alcove. Thus ϕ_i is an isometry taking the Weyl alcove (and the simplex) to itself.

To see that $\phi_i(\lambda \otimes \gamma) = \phi_i(\lambda) \otimes \gamma$, note that for any element σ of the quantum Weyl group $\sigma' \circ \phi_i = \phi_i \circ \sigma$, where σ' is another element of the quantum Weyl group with the same sign, because this is true for generating reflections. Thus in Formula (6)

$$N^{\eta}_{\lambda,\gamma} = \sum_{\sigma \in \mathfrak{W}_0} (-1)^{\sigma} m_{\gamma}(\lambda - \sigma(\eta)) = \sum_{\sigma \in \mathfrak{W}_0} (-1)^{\sigma} m_{\gamma}(\tau_i(\lambda - \sigma(\eta)))$$

$$= \sum_{\sigma \in \mathfrak{W}_0} (-1)^{\sigma} m_{\gamma}(\phi_i(\lambda) - \phi_i(\sigma(\eta))) = \sum_{\sigma' \in \mathfrak{W}_0} (-1)^{\sigma'} m_{\gamma}(\phi_i(\lambda) - \sigma'(\phi_i(\eta)))$$

$$= N^{\phi_i(\eta)}_{\phi_i(\lambda),\gamma}.$$

This confirms $\phi_i(\lambda \otimes \gamma) = \phi_i(\lambda) \otimes \gamma$.

Now ϕ_i certainly takes a neighborhood of the weight 0 to a neighborhood of the weight $k\lambda_i$, and since it is an isometry of the simplex it also connects these neighborhoods intersected with the simplex. Conversely, if λ is an extreme point of the simplex and ϕ is an isometry of a neighborhood of the weight 0 intersected with the simplex to a neighborhood of λ intersected with the simplex, then ϕ extends to an isometry from an entire neighborhood of the weight 0 to a neighborhood of the weight λ, which takes the hyperplanes $\{\gamma : (\alpha_j, \gamma) = 0\}$ to the hyperplanes $\{\gamma : (\alpha_j, \gamma) = 0\}$ for all $j \neq i$ and for some i together with the hyperplane $\{\gamma : (\theta, \gamma) = (\theta, \lambda)\}$. Therefore $\gamma \mapsto \phi_i(\gamma) - \lambda$ is an isometry taking 0 to 0 and the hyperplanes

$$\{\{\gamma : (\alpha_j, \gamma) = 0\} : 0 \leq j \leq r\}$$

to

$$\{\{\gamma : (\alpha_j, \gamma) = 0\} : j \neq i\} \cup \{\{\gamma : (\theta, \gamma) = 0\}\}.$$

Such an isometry is necessarily a composition $\tau\tau'$ with τ an element of the classical Weyl group and τ' a linear isometry permuting the simple roots $\{\alpha_j\}$ (such isometries of the Weyl chamber correspond to automorphisms of the Coxeter graph). The map τ corresponds to the base $\{\alpha_j\}_{j \neq i} \cup \{-\theta\}$, and therefore by Lemma 1, the associated λ_i is in the range of ℓ. Since the composition of isometries $\phi \circ (\tau')^{-1}\phi_i^{-1}$ of the alcove takes γ to $\gamma - k\lambda_i + \lambda$, we must have that $\lambda = k\lambda_i$ and we see that every corner of the simplex which is locally isometric to the corner 0 is in the image of $k\ell$.

To see that $k\ell$ is a homomorphism, let us first consider the case $k = 1$. Then if $\ell(z_1) = \lambda_i$ we have $\ell(z_1) \otimes \ell(z_2) = \phi_i(0) \otimes \ell(z_2) = \phi_i(\ell(z_2))$ which is again an extreme point locally isometric to 0, and thus is $\ell(z_3)$ for some z_3. Of course in general each η such that $N_{\lambda,\gamma}^{\eta} \neq 0$ differs from $\lambda + \gamma$ by an element of the root lattice, and thus $e_\eta = e_{\lambda+\gamma} = e_\lambda \cdot e_\gamma$. In particular $e_{\ell(z_3)} = e_{\ell(z_1 z_2)}$ and $z_3 = z_1 z_2$. Thus ℓ is an endomorphism.

Finally notice that the map multiplication by k where it is defined forms a commuting square with the maps ϕ_i acting respectively on the Weyl alcove at level 1 and the Weyl alcove at level k and thus the map k takes on the one hand $\ell(z_1 z_2)$ to $k\ell(z_1 z_2)$ and on the other $\ell(z_1) \otimes \ell(z_2) = \phi_i(\ell(z_2))$ to $\phi_i(k\ell(z_2)) = k\ell(z_1) \otimes k\ell(z_2)$ and thus $k\ell$ is an endomorphism. $\qquad\square$

Figure 3 illustrates the notions discussed above for rank one and two Lie algebras. The Weyl alcove for $k = 4$ is shown. Invertible weights are marked with a triangle (including the weight 0, which is labeled), other corners with a square. The lines of reflection generating the quantum Weyl group are indicated by dotted lines, and the root θ is marked. The elements of the root lattice are indicated by solid dots, squares and triangles and weights not in the root lattice by open figures.

A_1 $\qquad\qquad$ A_2 $\qquad\qquad$ B_2 $\qquad\qquad$ G_2

FIGURE 3. Weyl alcove and corners for low rank Lie algebras

Proof of Lemma 1. Suppose α_i is long, and $(\lambda_i, \theta) = 1$. Since (λ_i, θ) is the coefficient of α_i in the expansion of θ in terms of simple roots, we have
$$\theta = \alpha_i + \sum_{j \neq i} k_j \alpha_j$$
for some positive integers k_j. To conclude the existence of a unique τ_i as in the statement of the lemma, it suffices to show that $\{\alpha_j\}_{j \neq i} \cup \{-\theta\}$ is a base Since the coefficient of α_i in the expansion of θ is 1, every positive root can be written in terms of the original base with the

coefficient of α_i being either 0 or 1. In the former case the root is already a nonnegative combination of $\{\alpha_j\}_{j \neq i}$, in the second $\alpha = \theta + (\alpha - \theta)$ writes the positive root as a sum of two terms, each with all nonpositive coefficients in the new base.

That the second condition implies the first is a straightforward inverting of the above argument.

For the second assertion, notice that the homomorphism is well-defined on Λ/Λ_r, because $(\lambda_i, \alpha_j) = \delta_{i,j}$, so (λ_i, α) is an integer for all roots α. A count of such λ_i from Table 2 of [10][Chapt. 12] (For A_l all fundamental weights, for B_l λ_1, for C_l λ_l, for D_l λ_1, λ_{l-1} and λ_l, for E_6 λ_1, λ_6, and for E_7 λ_7) shows that we need only check that no λ_i gets sent to the trivial homomorphism. This is confirmed by computing inner products (λ_i, λ_j) using Table 1 of [10][Chapt. 13]. $\square$

2.3. **Degenerate objects are invertible.** Recall that θ represents the unique long root in the Weyl chamber. Let us use β to represent the unique short root in the Weyl chamber.

Proposition 3. *Let Z be a subgroup of $Z(G)$. Let Γ_Z be the set of weights in Λ_0 which are annihilated by $\chi_z = e_{\ell(z)}$ for all $z \in Z$. Notice Γ_Z is the sublattice of weights corresponding to classical representations of G_Z, intersected with Λ_0. Let Δ_Z be $k\ell[Z]$. Then Γ_Z and Δ_Z are closed subsets of Λ_0.*

Proof. By Equation (6), the weights occurring in the decomposition of the truncated tensor product of two weights lie in the product of their cosets in Λ/Λ_r, and thus are annihilated by any homomorphism which annihilates the factors (since the reflections that generate the quantum Weyl group preserve these cosets). Thus Γ_Z is closed under the truncated tensor product. Of course it is closed under the duality relation, since duality corresponds to inverse in Λ/Λ_r.

In Δ_Z the truncated tensor product and dual on simple objects corresponds to product and inverse in Z, so closure is immediate. $\square$

Remark 4. The sets Γ_Z we may view as classical closed sets. The closed subsets of the Weyl chamber are the restriction of sublattices of Λ containing the root lattice to the Weyl chamber, and the subsets Γ_Z are just the restriction of these to the alcove. the sets Δ_Z, however, have no correspondence to anything classical. These two classes of closed subset almost, but not quite, exhaust the list. It is shown in [24] that when (and only when) $k = 2$ there are certain additional closed subsets.

To address the question of whether a subset of the form Γ_Z or Δ_Z yields a modular category via the quotient of Section 1, we need to identify the degenerate objects of these sets. Since we have an explicit formula for C_λ, this is largely a matter of using Equation (6) to give a careful description of the truncated tensor product.

Lemma 2. *For any σ in the classical Weyl group $\mathfrak{W}$, and any weights γ, λ in the Weyl alcove, if $\lambda + \sigma(\gamma)$ is in the Weyl alcove, then $\lambda \otimes \gamma$ contains $\lambda + \sigma(\gamma)$ as a summand with multiplicity one.*

Proof. See end of Section 2.3. □

Lemma 3. *$\lambda \otimes \lambda^\dagger$ contains θ as a summand if $k \geq 2$ and λ is not a corner (i.e. a multiple of a fundamental weight such that $(\lambda, \theta) = k$). In the nonsimply-laced case it contains β as a summand unless $(\lambda, \alpha_i) = 0$ for every short simple root α_i.*

Proof. See end of Section 2.3. □

Theorem 4. *The set of degenerate objects of a closed subset of the form Γ_Z or Δ_Z is a set of the from $\Delta_{Z'}$ for some subgroup Z' of $Z(G)$. In particular, all degenerate objects are invertible.*

Proof. Of course since $\otimes$ on Δ_Z can be identified with group multiplication on Z, the only closed subsets of Δ_Z will be subgroups and hence of the form $\Delta_{Z'}$. So assume $\Gamma = \Gamma_Z$, and that $\lambda \in \Gamma$ is degenerate. We will show that λ is in the range of $k\ell$, which suffices for the theorem.

In that case if $k > 1$ then $\theta, \beta \in \Gamma$, so

$$C_\gamma C_\lambda^{-1} C_\theta^{-1} = 1$$

for any γ with $N_{\lambda,\theta}^\gamma \neq 0$, and likewise for β.

If λ is not a corner of the Weyl alcove then $N_{\lambda,\theta}^\lambda \neq 0$ by Lemma 3 so

$$C_\lambda C_\lambda^{-1} C_\theta^{-1} = \exp\left(\frac{-\pi i[(\theta,\theta) + 2(\theta,\rho)]}{k+h}\right) = \exp\left(\frac{-2\pi i h}{k+h}\right) \neq 1$$

so λ is not degenerate. Likewise in the nonsimply-laced case if $(\lambda, \alpha_i) \neq 0$ for some short simple root α_i then by Lemma 3 $N_{\lambda,\beta}^\lambda \neq 0$ so

$$C_\lambda C_\lambda^{-1} C_\beta^{-1} = \exp\left(\frac{-\pi i[(\beta,\beta) + 2(\beta,\rho)]}{k+h}\right) \neq 1$$

since $(\beta,\beta) + 2(\beta,\rho) < (\theta,\theta) + 2(\theta,\rho) = 2h$, so λ is not degenerate.

Now if $\lambda + \alpha$ is in the Weyl alcove for some long root α then by Lemma 2 $N_{\lambda,\theta}^{\lambda+\alpha} \neq 0$ so

$$C_{\lambda+\alpha} C_\lambda^{-1} C_\theta^{-1} = \exp\left(\frac{\pi i[2(\lambda,\alpha) + 2(\alpha,\rho) - 2(\theta,\rho)]}{k+h}\right)$$
$$= \exp\left(\frac{2\pi i[(\lambda+\rho,\alpha) - (\theta,\rho)]}{k+h}\right).$$

Also $|(\lambda+\rho,\alpha)| \leq k+h$ for all α, so $(\lambda+\rho,\alpha) - (\theta,\rho)$ can only be a multiple of $k+h$ if

1. $(\lambda+\rho,\alpha) = h - 1$ or
2. $(\lambda+\rho,\alpha) = -(k+1)$.

If λ is a corner and is orthogonal to all short simple roots, then λ is $k\lambda_i/n$ for α_i long, where $(\lambda_i,\theta) = n$. If $k > n$ then $\lambda - \alpha_i$ is in the Weyl alcove, $N_{\lambda,\theta}^{\lambda-\alpha_i} \neq 0$, and

$$(\lambda+\rho,-\alpha_i) = -k/n - 1.$$

For λ to be degenerate requires this quantity to be equal to $-k - 1$ (since it is negative, it is not $h - 1$), so that $n = 1$ and we conclude λ is in the range of $k\ell$ by Lemma 1.

Thus the only possible degenerate objects which are not in the range of $k\ell$ are weights λ_i dual to long roots for $k = (\lambda_i,\theta)$. We argue first that for each such λ_i there is a long positive root α with $(\lambda_i,\alpha) = 0$ such that either $\lambda_i + \alpha$ is in the Weyl alcove with $k = (\lambda_i,\theta)$ or $\alpha - \theta$ is a long root and $\lambda_i + \alpha - \theta$ is in the Weyl alcove, and second that this contradicts degeneracy.

To see the existence of such an α, observe from the Dynkin diagrams [10][pg. 58] that for every fundamental weight λ_i dual to a long root, either $(\lambda_i,\theta) = 1$, $\lambda_i = \theta$, or for one of the subdiagrams into which the removal of λ_i divides the diagram, the weight λ_j adjacent to λ_i is dual to a long root and satisfies $(\lambda_j,\theta') = 1$ for θ' the highest root associated to this subdiagram. In the first case λ_i is in the image of $k\ell$, in the second $\alpha = \iota$ will do, and in the third we choose $\alpha = \theta'$. Of course in the third case $(\alpha,\lambda_i) = 0$, $(\alpha,\lambda_k) \geq 0$ for $k \neq i$, and $(\lambda_i + \alpha, \alpha_i) = -1 + 1$ because the decomposition of α into simple roots contains exactly one simple root adjacent to α_i. So $\lambda_i + \alpha$ is in the Weyl chamber. If $(\alpha_i,\theta) = 0$, then $\lambda_i + \alpha$ is in fact in the Weyl alcove. If not then $(\alpha,\theta) = 1$. Except for the case A_l, where all corners are in the range of $k\ell$ and there is nothing to prove, θ is of the form λ_k for some k, so $(\alpha,\theta) = 1$ indicates that the subdiagram contained that λ_k. Further inspection of the Dynkin diagrams indicates that the λ_i for which the only subdiagram meeting the desired conditions contains

this λ_k are λ_2 of E_7 and λ_1 of E_8. In the first case $(\alpha, \alpha_k) = 1$ and thus $\lambda_2 + \alpha - \theta$ is in the Weyl alcove. In the second use $\alpha + \alpha_8$ is a positive root, $\alpha + \alpha_8 - \theta$ is a root, and $\lambda_i + \alpha + \alpha_8 - \theta$ is in the Weyl alcove, so $\alpha + \alpha_8$ meets the desired condition.

Thus we need only check that the existence of such α contradicts degeneracy. In the first case

$$0 < (\lambda + \rho, \alpha) = (\rho, \alpha) < h - 1,$$

because of course $\alpha \neq \theta$. In the second

$$0 > (\lambda + \rho, \alpha - \theta) = -k + (\rho, \alpha) > -k - 1.$$

Thus the only degenerate objects for Γ are those in $k\ell[Z(G)]$.

Now if $k = 1$, it is not true that $\theta \in \Gamma$. However, the only elements of Λ_0 and hence of Γ are elements of $k \circ \ell[Z(G)]$ and weights λ_i with α_i short. Thus the argument involving C_β above suffices to show that only those in $k\ell[Z(G)]$ can be degenerate. $\square$

Corollary 1. *Every closed subset of the form Γ_Z or Δ_Z yields, via the quotient of Section 1, a modular category whose associated TQFT and invariant is as described in Section 1 and the Appendix.*

Proof. Since we have exempted D_{2n}, Z is cyclic. $\square$

Proof of Lemma 2. We note first that if λ is in the Weyl alcove, and σ is an element of the quantum Weyl group taking a weight μ not in the Weyl alcove into the Weyl alcove, then the distance between λ and $\sigma(\mu)$ is strictly less than the distance between λ and μ. To see this, note that if $\sigma(\mu)$ is in the Weyl alcove, than μ cannot lie on one of the 'walls of the Weyl alcove,' i.e. the hyperplanes reflection about which generates the quantum Weyl group (if it did, it and all its conjugates would have nontrivial stabilizers, which is not true of any point in the Weyl alcove). In that case, one of the walls of the alcove lies between λ and μ, and thus reflection about this wall brings μ strictly closer to λ. repeating this procedure brings a sequence of weights conjugate to μ getting strictly closer to λ. Since there are only finitely many weights in the weight lattice a given distance from λ, this process must end after finite time. It can only end by reaching a point which is conjugate to μ and which lies in the alcove or on the walls. Since the quantum Weyl group acts transitively on the Weyl alcove [12], this gives the claim.

Now let λ, γ, and σ be as in the statement of the lemma, so any μ for which $m_\gamma(\lambda - \mu)$ is nonzero must be a distance at most $||\gamma||$ from λ, so if μ is not in the Weyl alcove but is conjugate to μ' which is, then μ' is a distance less than $||\gamma||$ from λ, and hence is not $\lambda + \sigma(\gamma)$. Thus

the only contribution to $N_{\lambda,\gamma}^{\lambda+\sigma(\gamma)}$ in Formula (6) comes from $\sigma = 1$, and would be $m_\gamma(\sigma(\gamma)) = 1$. $\qquad\square$

Proof of Lemma 3. We will make the argument for θ, noting parenthetically how it differs for β when not simply-laced. We will actually prove that $N_{\lambda,\theta}^\lambda$ is nonzero, which is equivalent. Recall that since ι is the weight 0 $m_\theta(\iota) = r$ (respectively $m_\beta(\iota) = r_0$, the number of short simple roots). Thus in the sum (6), there is a contribution of r (r_0) from the identity element of the quantum Weyl group and a contribution for each σ such that $m_\theta(\lambda, \sigma(\lambda)) \neq 0$. By the first paragraph in the proof of Lemma 2 above, we saw that σ can be written as a product of reflections each taking λ strictly farther away from itself. Since $\sigma(\lambda) - \lambda$ is in the root lattice, we conclude that after one such reflection its length is at least that of a short root, after two its length is at least that of a long root, and after three it must be longer than a long root. Thus if σ is such a product of three or more reflections, $m_\theta(\lambda - \sigma(\lambda)) = 0$. A product of two reflections only increases $N_{\lambda,\theta}^\lambda$, so it suffices to consider the effect of a single reflection. If σ is reflection about one of the walls of the Weyl alcove and $m_\theta(\lambda - \sigma(\lambda)) \neq 0$ $(m_\beta(\lambda - \sigma(\lambda)) \neq 0)$ then $\lambda - \sigma(\lambda)$ is either $-\alpha_i$ or θ, depending on which wall, and $m_\theta(\lambda - \sigma(\lambda)) = 1$ $(m_\beta(\lambda - \sigma(\lambda)) = 1$ if the root is short). Thus $N_{\lambda,\theta}^\lambda$ is at least r minus the number of walls of the Weyl alcove to which λ is adjacent (r_0 minus the number of walls dual to short roots to which λ is adjacent). In the simply-laced case, only corners are adjacent to r walls. In the nonsimply-laced case, only weights for which $(\lambda, \alpha_i) = 0$ for all short simple roots α_i are adjacent to r_0 walls dual to a short root. $\qquad\square$

2.4. **TQFTs from closed subsets.**

Lemma 4. *If $k\lambda_i$ is invertible and γ is any weight, then*

$$S_{k\lambda_i,\gamma}/\operatorname{qdim}(\gamma) = C_{\phi_i(\gamma)}C_{k\lambda_i}^{-1}C_\gamma^{-1} = e^{2\pi i(\lambda_i,\gamma)}$$

where $\phi_i(\gamma)$ is the simple weight $k\lambda_i\otimes\gamma$.

Proof. First, we note that $\tau_i^{-1}(\lambda_i) = -\lambda_i$, because they have the same inner product with the simple roots, and that $(\rho, \gamma - \tau_i(\gamma)) = h(\lambda_i, \gamma)$ for all γ. It suffices to check the second claim on simple roots. For $j \neq i$, we have that $\tau_i(\alpha_j)$ is another simple root of the same length, different from α_i, and thus both sides of the equation are zero (the object ι). For α_i, we have $\tau_i(\alpha_i) = -\theta$, so $(\rho, \alpha_i - \tau_i(\alpha_i)) = (\rho, \theta) + 1 = h = h(\lambda_i, \alpha_i)$.

By Equation (7)

$$C_{\phi_i(\gamma)}C_{k\lambda_i}^{-1}C_\gamma^{-1} =$$
$$\exp\left(\frac{\pi i[(\tau_i(\gamma),\tau_i(\gamma)) - (\gamma,\gamma) + 2(\rho,\gamma - \tau_i(\gamma)) - 2k(\tau_i(\gamma),\lambda_i)]}{k+h}\right),$$

where we have used the fact that $\phi_i(\gamma) = k\lambda_i - \tau_i(\gamma)$. Using the fact that τ_i is an isometry and the identities in the previous paragraph gives

$$C_{\phi_i(\gamma)}C_{k\lambda_i}^{-1}C_\gamma^{-1} = \exp\left(\frac{2\pi i}{k+h}[h(\lambda_i,\gamma) + k(\lambda_i,\gamma)]\right) = e^{2\pi i(\lambda_i,\gamma)}. \quad \square$$

For the rest of this article let Γ be a closed subset of the Weyl alcove of the form Γ_Z or Δ_Z.

Theorem 5. *The degenerate invertible elements of Γ are exactly the invertible elements of Γ which, viewed under $e \circ k^{-1}$ as a subgroup of $(\Lambda/\Lambda_r)^*$, annihilate the image of Γ in Λ/Λ_r. In particular if Γ is Γ_Z for some Z, then the degenerate invertible elements are those in the image of Z under $k\ell$ intersected with Γ. These are even or odd depending on whether $k(\lambda_i,\lambda_i)$ is an even or odd integer.*

Proof. If $k\lambda_i$ is invertible, Müger shows it is degenerate for Γ if and only if $S_{k\lambda_i,\gamma} = \mathrm{qdim}(\gamma)$ for every $\gamma \in \Gamma$, which by Lemma 4 is true if and only if e^{λ_i} annihilates all $\gamma \in \Gamma$.

To determine whether $k\lambda_i$ is odd or even, we need to check whether $C_{k\lambda_i} = \pm 1$, which is to say whether

$$k(k(\lambda_i,\lambda_i) + 2(\rho,\lambda_i))$$

is an even or odd multiple of $k + h$. Notice $2(\rho,\lambda_i) = (\rho,\lambda_i - \tau_i(\lambda_i)) = h(\lambda_i,\lambda_i)$ (Using the identities in the proof of the previous lemma). So we are asking whether $k(\lambda_i,\lambda_i)$ is an even or odd integer. $\square$

Corollary 2. *For every $\mathfrak{g}$ and for every level k, the ribbon category associated to the full Weyl alcove is modular.*

Corollary 3. *If $\Gamma = \Gamma_Z$ and k is one of the levels conjectured by Dijkgraaf and Witten to admit a Chern-Simons theory for G, i.e., if $k(\ell(z),\ell(z))/2$ is an integer for each z in Z, then in the ribbon category associated to Γ all degenerate objects are even invertible elements, and in fact form a group isomorphic to Z. Thus as in Section 1 Γ yields a modular category and a TQFT. These levels are exactly the levels at which Z embeds into Γ_Z as even degenerate objects via the map $k\ell$.*

3. Products of Modular Categories and Tensor Products of TQFTs

At this point we have technically succeeded in the goal of the paper: We have constructed TQFTs associated to nonsimply-connected groups at the levels predicted by physics. But in fact we have an embarrassment of riches, in that we have constructed many more TQFTs than that. First of all there are the closed subsets Δ_Z of the Weyl alcove consisting entirely of invertible elements, which at many values of k give TQFTs by the method of Section 1. Second, the levels k suggested by Dijkgraaf and Witten are not the only ones giving TQFTs by any means: While these authors suggest levels which are a *multiple* of a certain N, it is easy to check from Theorem 5 that Γ_Z is modular (without the need for a quotient) whenever k is *relatively prime* to N. More generally, Γ_Z gives a TQFT whenever it contains no odd degenerate objects. It is incumbent upon us to give as complete as possible a description of these 'unexpected' theories and, if we claim to have verified the expectations of physics, to show that they contain no new, nontrivial information in some sense. This is the goal of this section.

3.1. **Products of modular categories.** Suppose that Γ is the label set of a ribbon category, and Γ contains two closed subsets Γ' and Γ'' such that

1. the intersection $\Gamma' \cap \Gamma''$ consists of even degenerate objects,
2. the product $\otimes$ of any element of Γ' with an element of Γ'' is simple (i.e. is an element of Γ),
3. every element of Γ is a product of an element of Γ' and Γ'' and
4. if $\lambda' \in \Gamma'$ and $\lambda'' \in \Gamma''$ then $C_{\lambda' \otimes \lambda''} = C_{\lambda'} C_{\lambda''}$.

Then we say that Γ is the *product* of Γ' and Γ''.

Notice that $R_{\lambda',\lambda''} = R_{\lambda'',\lambda'}^{-1}$ because of Condition 4. In particular, consider the invariant of a link with components labeled by elements of Γ. Every label can be written as a product of a label in each of the subsets (not uniquely, but since different choices will disagree by a factor of an even degenerate object, it will not effect the argument to follow), and using the tensor product property of the link invariant, it can be written as the invariant of a link with twice as many components, all labeled by elements of one of the two subsets. Now because of the condition on the R-matrix, this invariant is equal to the invariant of two unlinked copies of the original link, one labeled by labels in Γ' and one by labels in Γ''. In this sense the invariant associated to Γ is the product of the invariants associated to the two factors. This is the motivation for the definition.

The S-matrix for Γ is the tensor product of the S-matrices of the factors, so Γ corresponds to a modular category if and only if the factors do.

Proposition 4. *Suppose $Z \subset Z'$ are subgroups of the center of G, Γ' is the closed subset generated by $\Gamma_{Z'}$ and Δ_Z, and $Z_0 = Z \cap (k\ell)^{-1}[\Gamma_{Z'}]$. Then $\Gamma' \subset \Gamma_{Z_0}$ is of the form Γ_Y for some Y and $\Delta_{Z_0} = \Delta_Z \cap \Gamma_{Z'}$ consists of degenerate invertibles for Γ'. If all of Δ_{Z_0} is even then Γ' is the product of $\Gamma_{Z'}$ and Δ_Z. These are the only cases in which Γ decomposes into a product, apart from D_{2n}. In the case $Z = Z'$, $\Gamma' = \Gamma_{Z_0}$ is one of the closed subsets described in Corollary 3.*

Proof. Since $Z_0 \subset Z$, the image of Z_0 under χ annihilates Γ_Z and hence $\Gamma_{Z'}$. Since $Z_0 \subset (k\ell)^{-1}[\Gamma_{Z'}]$, $\Delta_{Z_0} \subset \Gamma_{Z'}$ so the image of Z under χ annihilates Δ_{Z_0}, or equivalently the image of Z_0 annihilate Δ_Z. Thus the image of Z_0 annihilates Γ'. By Lemma 4, this means Δ_{Z_0} (which of course is contained in Γ') will consist of degenerate units for Γ'.

If Δ_{Z_0} consists entirely of even degenerate objects, then Condition 1 is met. Conditions 2 and 3 are clear, and Condition 4 follows from Lemma 4. In the case where $Z = Z'$, notice Γ' contains $\Lambda_0 \cap \Lambda_r$. Thus by Lemma 2 Γ' consists of a union of cosets of Λ_0/Λ_r, which is to say that it is of the form Γ_Y for some Y. Of course $Y \supset Z_0$, but anything in Y annihilates Γ_Z and Δ_Z, so it is contained in Z and in $(k\ell)^{-1}[\Gamma_Z]$, so $Y = Z_0$ and $\Gamma' = \Gamma_{Z_0}$. By Corollary 3, this is one of the Dijkgraaf-Witten theories.

To see these are the only cases of products of ribbon categories, suppose Γ is a closed subset which can be written as a product of two closed subsets Γ' and Γ''. If both were of the form Γ_Z, and not of the form Δ_Z, then both would contain θ. Since this is not invertible except in the case su_2 at level 2, when one would have to be $\Delta_{\mathbb{Z}_2}$, Condition 1 prevents both from being only of the form Γ_Z. They cannot both be of the form Δ_Z, because then the product would be of the form Δ_Z with a Z which was a product of groups, which only happens for $\mathfrak{g} = D_{2n}$.

Thus one must be of the form Δ_Z and one of the form $\Gamma_{Z'}$ for some Z, Z'. But by Condition 4 and Lemma 4, Z must annihilate $\Gamma_{Z'}$ under χ, and thus $Z \subset Z'$. $\qquad\square$

3.2. **Tensor product of TQFTs.**

Proposition 5. *If the degenerate objects of $\mathfrak{C}$ are all even and invertible and form a cyclic group, and if the label set Γ is a product of Γ' and Γ'', then the modular quotient of $\mathfrak{C}$ is the product of the images in that quotient of Γ' and Γ''.*

Proof. A little thought will convince the reader that this result is almost immediate assuming that the intersection (a group of degenerate invertibles) acts freely on Γ. This is in fact the case, though one would like a more direct argument than the one below. Let Z_0 be the group $\Gamma' \cap \Gamma''$.

Recall that, if V is the vector space of formal linear combinations of elements of Γ, then one can extend the link invariant to an invariant of links labeled by V in such a way that it is linear in each component. Furthermore, the functor to $\mathfrak{C}'$ gives a linear map from V into the corresponding V' which is consistent with the link invariant and a vector is in the kernel of this map if and only if it is in the kernel of the invariant for every link. In particular the Hopf link gives a nondegenerate pairing on the image of V in V'. Bruguières proves that if the Hopf link is labeled respectively by $\lambda \in \Gamma$ and $\omega = \sum_{\gamma \in \Gamma} \operatorname{qdim}(\gamma)\gamma$ then the invariant is 0 unless λ is degenerate, in which case it is $\operatorname{qdim}(\lambda)\operatorname{qdim}(\omega) = \operatorname{qdim}(\omega)$. The same is necessarily true for $\lambda' \in \Gamma'$ and $\omega' = \sum_{\gamma' \in \Gamma'} \operatorname{qdim}(\gamma')\gamma'$ and likewise for λ'', ω'' in Γ''. Thus the value of a Hopf link labeled by $\omega' \otimes \omega''$ and a typical element $\lambda' \otimes \lambda''$ in Γ is zero unless λ' and λ'' are both degenerate. In particular, since of course it is nonzero if $\lambda' \otimes \lambda''$ is degenerate, every degenerate object in Γ can be written as a product of degenerate elements in Γ' and Γ'', so in fact $\omega' \otimes \omega''$ gives zero on the Hopf link exactly when it is paired with a nondegenerate simple object. Thus a multiple of $\omega' \otimes \omega''$ gives the same functional on V via the Hopf link as ω. Since the Hopf link labeled by ω and ω gives $\operatorname{qdim}(\omega)|Z|$ and that labeled by ω and $\omega' \otimes \omega''$ gives $\operatorname{qdim}(\omega)|Z| \cdot |Z_0|$, we conclude that $\omega' \otimes \omega'' = |Z_0|\omega$, and thus that Z_0 acts freely on Γ.

With this in hand, if $\lambda = \lambda' \otimes \lambda''$ then the stabilizer of λ is the product of the stabilizers of λ' and λ''. The image of λ' and λ'' in $\mathfrak{C}'$ is a sum of as many simple objects as there are elements in the stabilizer, and since each of these is a direct summand in the image of λ, they must each be simple. Every simple object in $\mathfrak{C}'$ arises this way, so every simple object in $\mathfrak{C}'$ is the product of an object in the image of Γ' and Γ''. Since $\mathfrak{C}'$ contains no degenerate objects, it is necessarily a product of these two subcategories. $\qquad\square$

In Appendix A.2, we define the notion of the tensor product of two TQFTs and show that the product of two modular categories gives the tensor product of their two TQFTs (that the invariant is the product of the two invariants is an easy consequence of the fact that the link invariant is the product of the link invariants. From this and the previous subsection we can deduce all the tensor product relationships between the TQFTs we have constructed.

Corollary 4. *In the situation described in Proposition 4, the three TQFTs are related by $\mathfrak{Z}_{\Gamma'} = \mathfrak{Z}_{\Delta_Z} \otimes \mathfrak{Z}_{\Gamma_{Z'}}$, and (except for $\mathfrak{g} = D_{2n}$) every TQFT arising from some Γ_Z can be tensored with one arising from Δ_Z to give one of the TQFTs conjectured by Dijkgraaf and Witten.*

Remark 5. If we think of the product $\otimes$ as analogous to an algebra product, then to say of two objects γ and λ that $R_{\gamma,\lambda} = R_{\lambda,\gamma}^{-1}$ is analogous to saying that they are commuting elements of the algebra. Thus we should think of the subcategory of degenerate objects as being the 'center' of the ribbon category, and the quotient of Müger and Bruguières is the quotient of an algebra by its center to give an algebra with a trivial center (modularity). Extending this, the definition of product is to say that the algebra is generated by two commuting subalgebras, and thus the quotient is a product of their quotients.

3.3. **Modular categories consisting entirely of invertibles.** It remains to identify the TQFTs $\mathfrak{Z}_{\Delta_Z}$ where Z is a subgroup of the center $Z(G)$. In fact the resulting TQFT and three-manifold invariant have already been defined and studied by Murakami, Ohtsuki and Okada [20].

Proposition 6. *If $\mathfrak{g} \neq D_{2n}$ the three-manifold invariant arising from Δ_Z in the case when Δ_Z has no odd degenerate elements is*

$$(8) \qquad Z_N(M,r) = \left(\frac{G_N(r)}{|G_N(r)|} \right)^{-\sigma(A)} |G_N(r)|^{-n} \sum_{l \in (\mathbb{Z}_2)^n} r^{l^t A l}$$

where N is the order of Δ_Z modulo the even degenerate objects, A is the linking matrix of a framed link presenting the three-manifold M, n is the number of components, $\sigma(A)$ is the signature of A, l^t is the transpose of the vector l and $G_N(r) = \sum_{m=1}^{N} r^{m^2}$ where r is $\exp(k\pi i(\lambda_i, \lambda_i))$, with λ_i the element of $\ell[Z]$ giving $k(\lambda_i, \lambda_i)$ the largest denominator. This is the invariant constructed by Murakami, Ohtsuki and Okada.

Proof. Of course it suffices to see that, with I as defined in Section 1.3, $I(L)$ is $\sum_{l \in (\mathbb{Z}/N)^n} r^{l^t A l}$.

Let λ be a generator of Δ_Z. then λ^N is even degenerate and no smaller power of λ is degenerate. To say that λ^n for some n is degenerate for Δ_Z is to say that $C_{\lambda^{n+1}} = C_{\lambda^n} C_\lambda$, since it suffices to check the degeneracy condition against a generator. Now $\lambda = k\lambda_i$ and $\lambda^n = k\lambda_j$ for some $\lambda_i, \lambda_j \in \ell[Z]$, so by Lemma 4, this is to say that $k(\lambda_i, \lambda_j)$ is an integer, which by the fact that the map e is a homomorphism is equivalent to saying $kn(\lambda_i, \lambda_i)$ is an integer. Thus for any λ_j the

denominator of $k(\lambda_j, \lambda_j)$ represents the order of $k\lambda_j$ in the quotient group Δ_Z modulo even degenerates, and thus the statement that λ is a generator is equivalent to $k(\lambda_i, \lambda_i)$ having maximal denominator as in the statement of the proposition. Let $r = \exp(\pi i k(\lambda_i, \lambda_i))$. Notice $r^{2N} = 1$, and N is the least natural number for which this is true.

The fact that λ^N is even degenerate means that $C_{\lambda^N} = 1$. Applying Lemma 4 recursively shows $C_{\lambda^N} = C_\lambda^N \exp(\pi i k N(N-1)(\lambda_i, \lambda_i))$, while of course $C_\lambda = \exp(\pi i(k\lambda_i, k\lambda_i + 2\rho)/(k + h)) = \exp(\pi i k(\lambda_i, \lambda_i)) = r$, so $C_{\lambda^N} = r^{N^2}$. In order for this to be 1 we conclude that N is odd and r is a primitive Nth root of unity or N is even and r is a primitive $2N$th root of unity. We claim that the link invariant of a link with linking matrix A and with the n components labeled by the vector of labels $l \in (\mathbb{Z}/N)^n$ or $l \in (\mathbb{Z}_{2N})^n$ depending on the parity of N, where we mean labeling the ith component by λ^{l_i}, is $r^{l^t A l}$.

To see this, notice that because all labels are units $R_{\lambda^n, \lambda^m} R_{\lambda^m, \lambda^n} = S_{\lambda^n, \lambda^m} \cdot \mathrm{id} = r^{2nm}$, which means that switching an undercrossing to an overcrossing in a link projection with these labels multiplies the link invariant by r^{2nm}, which is exactly the effect this move has on $r^{l^t A l}$. Since such moves will untie any link to a collection of unlinked framed unknots, it suffices to check that the formula is correct on these. This follows from the fact that both assign r^{mn^2} to the m-framed unknot with label n.

Now that we know the link invariant, $I(L)$ is either $\sum_{l \in (\mathbb{Z}_N)^n} r^{l^t A l}$ or $\sum_{l \in (\mathbb{Z}_{2N})^n} r^{l^t A l}$ depending on the parity of N. In the second case the $\mathbb{Z}_2$ symmetry of the labels because of the even degenerate unit means that this sum is $2^n \sum_{l \in (\mathbb{Z}_N)^n} r^{l^t A l}$ and as noted in Section 1.3 an overall factor in I depending on the number of components is canceled out in the three-manifold invariant. $\square$

Remark 6. The product of this invariant with its conjugate (or on the level of TQFTs, the tensor product of this TQFT with its conjugate TQFT) is an example of the finite group invariants constructed by Dijkgraaf and Witten in the same paper [6] as the one where they construct Chern-Simons theory from nonsimply-connected Lie groups. Thus Murakami et al's theories stand in the same relationship to the finite group theories as Chern-Simons theory does to Turaev-Viro (sometimes called three-dimensional gravity).

Appendix

A.1. **Constructing TQFTs from the ribbon $*$-category.** Recall the definition of a 2-framed three-dimensional TQFT given in [25],

an assignment of a vector space $\mathfrak{Z}(\Sigma_g)$ to a given oriented 2-framed surface Σ_g, and an assignment of a functional $\mathfrak{Z}(M)\colon \bigotimes_{i=1}^n \mathfrak{Z}(\Sigma_{g_i}) \to \mathbb{C}$ to each 2-framed three-manifold with boundary parameterized by an orientation reversing homeomorphism from $\bigcup_{i=1}^n \Sigma_{g_i}$ satisfying certain conditions.

Following [26] we construct a TQFT from a modular category as follows. Choose a handlebody H_g with boundary Σ_g, choose a framed graph in H_g whose image is a retract of H_g, and let $\mathfrak{Z}(\Sigma_g)$ be the set of all labelings of the graph, with two labelings identified if they give the same invariant for all embeddings (this is just the sum over all labelings of the edges of the tensor product over all vertices of the space of possible labelings of the vertex, as described in Section 1.1). If M is a 2-framed three-manifold with boundary parameterized by $\bigcup_{i=1}^n \Sigma_{g_i}$, we can present M by an embedding of $\bigcup_{i=1}^n H_{g_i}$ into S^3 together with a framed link in the complement of the embedding, with M homeomorphic to surgery on the link in the complement of the embedding by a map preserving the parameterization of the boundary. The functional $\mathfrak{Z}(M)$ evaluated on vectors corresponding to certain labelings of the n graphs in $\{H_{g_i}\}$ is the invariant of the labeled graphs embedded in S^3 together with the link, each component of the link being labeled by ω (times a normalization factor which need not concern us).

We will identify a labeling of a graph by data in the modular quotient $\mathfrak{C}'$ with a labeling of a slightly modified graph by data in the original category $\mathfrak{C}$ in such a way that the invariant of any ribbon graph labeled by data in the modular quotient is the same as the invariant of the modified graph with the associated data in the original graph (actually, it will be a formal linear combination of such labelings). This will tell us how to construct the vector spaces $\mathfrak{Z}(\Sigma_g)$ and the functionals $\mathfrak{Z}(M)$ from data in $\mathfrak{C}$ (we have already seen that links labeled by ω in $\mathfrak{C}'$ can be replaced by the same link labeled by ω in $\mathfrak{C}$ up to a factor, and that continues to be true in the presence of other graph components).

Given a graph G, the modified graph will be formed by adding a new edge to each vertex of G, with all the new edges meeting in a single new vertex. Of course an embedding of G into S^3 extends in many ways to an embedding of this larger graph, but since all the new edges will be labeled by degenerate objects in $\mathfrak{C}$, all these embedding will yield the same invariant and we are not obliged to specify a particular embedding.

Recall that Müger's construction of $\mathfrak{C}'$ proceeds in two steps. He first constructs an intermediate category $\mathfrak{C}^0$ whose objects are the objects of $\mathfrak{C}$ but whose morphisms from λ to γ are $\bigoplus_\mu \hom(\lambda, \gamma \otimes \mu)$, where the sum is over all degenerate simple objects $\mu \in \Gamma$. The category $\mathfrak{C}'$ is

constructed from this by the usual process of closing under subobjects. In particular $\mathfrak{C}^0$ is a full subcategory of $\mathfrak{C}'$ and for each object λ of $\mathfrak{C}'$ there is an object γ of $\mathfrak{C}^0$ and a morphism $f\colon \lambda \to \gamma$ such that f^*f is the identity and ff^* is a minimal idempotent.

In particular, if G is a graph labeled by data in $\mathfrak{C}'$ we can replace each object λ labeling an edge by the corresponding γ in $\mathfrak{C}^0$ as above, and compose each vertex label with the appropriate f and f^* morphisms to get a new labeling by data in $\mathfrak{C}^0$ with the same value of the invariant on each embedding. Thus we need only describe the process for a graph labeled by data in $\mathfrak{C}^0$.

So let G be an abstract graph labeled by data in $\mathfrak{C}^0$, and let G' be the extended version of G above. Each edge of G is labeled by an object of $\mathfrak{C}^0$, which is also an object of $\mathfrak{C}$, so we label the corresponding edge of G' by that object. Assume that each new edge in G' was added so as to be last in the ordering of edges around its vertex. Then the label of a given vertex is an element of $\bigoplus_\mu \hom(\lambda \otimes \mu^\dagger, \iota) \cong \bigoplus_\mu \hom(\lambda, \mu)$, where λ is the tensor product of the edge labels of G or their duals as in Section 1.1. We can assume that each label is in exactly one of these direct summands, as the general case can be expressed as a linear combination of such. So for each vertex there is a degenerate label μ such that the the label is $x \in \hom(\lambda, \mu)$. Label the new edge of G' by μ (oriented away) and the vertex by x. The new vertex needs a label in $\hom(\mu_1 \otimes \cdots \otimes \mu_n, \iota)$, which of course is the identity morphism if $\mu_1 \otimes \cdots \otimes \mu_n = \iota$ and 0 otherwise. It is now a simple exercise to check that for any embedding of G and any extension of that to an embedding of G', the invariant of G equals the invariant of G'.

Two other observations follow by simple calculations. The first is that we could as well choose G' to have only one new edge for each connected component, the cost being the labels on the edges of G might have to be multiplied by degenerate objects. Thus the vector space $\mathfrak{Z}(\Sigma_g)$ is still spanned by labelings of a graph in H_g which is a retract of H_g, but it has an extra edge (which is required to be labeled by a degenerate object) and an extra univalent vertex.

In particular the vector space of the torus, which can be viewed as the space of labels for the link invariant, will have a basis element for each orbit under Z of simple objects on $\mathfrak{C}$, as well as a basis element for each such object and any nontrivial element of its stabilizer, the stabilizing element labeling the extra edge. These basis elements with nontrivial labels on the extra edge give a nonzero link invariant only when there is another component in the link which is labeled by a basis element with a nontrivial label. A similar picture appears in the literature

in descriptions of WZW models arising from nonsimply-connected Lie groups [18, 9].

Note that each of the vector spaces associated to surfaces decomposes naturally as a sum of sectors according to the label of the extra edge. This label is an element of the group Z, which in the nonsimply-connected groups we explore in Section 2 represents the fundamental group of the group, which also indexes different principal bundles of the group G_Z over a connected surface. Thus we may view the vector spaces $\mathfrak{Z}(\Sigma_g)$ as a sum over contributions from the different principal bundles of the surface, as expected from the physics.

The second observation is that if the process of constructing G' with labels in $\mathfrak{C}$ is applied to a link component labeled by ω (think of this as one edge labeled by the direct sum of all simple representations and a single bivalent vertex labeled by the sum of the canonical duality map times the quantum dimension), the labeling morphisms lie in the trivial component of $\bigoplus_\mu \hom(\lambda, \mu)$, and thus we do not need to add the extra edge, and the result is the same link component labeled by ω is $\mathfrak{C}$. Thus although $\mathfrak{C}$ is not modular ω continues to be a perfectly good 'surgery label,' and the only adjustment that needs to be made to Reshetikhin and Turaev's process is an extension of the graphs spanning $\mathfrak{Z}(\Sigma_g)$.

A.2. **The product of two modular categories gives the tensor product of their TQFTs.** We say that a TQFT $\mathfrak{Z}$ is the tensor product of two TQFTs $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ if for each g there is an isomorphism $\Phi_g\colon \mathfrak{Z}(\Sigma_g) \to \mathfrak{Z}_1(\Sigma_g) \otimes \mathfrak{Z}_2(\Sigma_g)$ such that $\mathfrak{Z}(M) \circ \Phi^{-1} = \mathfrak{Z}_1(M) \otimes \mathfrak{Z}_2(M)$, where here we have identified the two domain spaces which are connected by a rearrangement of the tensor factors.

Proposition 7. *If $\mathfrak{C}$ is modular and has label set Γ which is a product of Γ' and Γ'', then the TQFT associated to $\mathfrak{C}$ is the tensor product of the TQFTs associated to the subcategories determined by Γ' and Γ''.*

Proof. Of course if $\mathfrak{C}$ is modular then $\Gamma' \cap \Gamma'' = \{\iota\}$ and it is easy to see that the category is a product of the two corresponding subcategories.

The central observation is that if G is any abstract graph and l is a choice of an object $\gamma_i \in \Gamma$ for each edge i and $x_j \in \hom(\lambda_j, \iota)$ for each vertex j, where λ_j is constructed out of $\{\gamma_i\}$ as in the definition of the graph invariant in Section 1.1, then there exists a set of pairs of labels l'_β and l''_β for some index β, with all the data in l' and l'' coming from the categories associated to Γ' and Γ'' respectively, such that for any embedding of G into S^3 the invariant of G labeled by l is the sum over all β of the product of the invariants of the embedding of G labeled by l'_β and l''_β respectively.

To see this, notice there are unique $\gamma_i' \in \Gamma'$ and $\gamma_i'' \in \Gamma''$ such that $\gamma_i = \gamma_i' \otimes \gamma_i''$. If we write $\lambda_j = \overline{\gamma}_{i_1} \otimes \overline{\gamma}_{i_2} \otimes \cdots \otimes \overline{\gamma}_{i_n}$, with $\overline{\gamma}_i = \gamma_i$ or $\gamma_i^\dagger$, and define $\lambda_j' = \overline{\gamma}_{i_1}' \otimes \cdots \otimes \overline{\gamma}_{i_n}'$ and $\lambda_j'' = \overline{\gamma}_{i_1}'' \otimes \cdots \otimes \overline{\gamma}_{i_n}''$, then there is a canonical isomorphism formed from a product of R-morphisms $\lambda_j' \otimes \lambda_j'' \cong \lambda_j$ because of the relation $R_{\gamma',\gamma''} = R_{\gamma'',\gamma'}^{-1}$. In particular there is an isomorphism

$$I \colon \hom(\lambda_j', \iota) \otimes \hom(\lambda_j'', \iota) \to \hom(\lambda_j, \iota)$$

so we can write

$$x_j = I(\sum_{\alpha_j} x_{j,\alpha_j}' \otimes x_{j,\alpha_j}'').$$

Now for a choice β of an α_j for each vertex j, we can define l_β' and l_β'' by $\{\gamma_i', x_{j,\alpha_j}'\}$ and $\{\gamma_i'', x_{j,\alpha_j}''\}$ respectively. Consider an embedding of G, and consider two parallel copies of that embedding of G one shifted from the other by the framing, and one labeled by l_β', the other by l_β''. On the one hand the fact that $R_{\gamma',\gamma''} = R_{\gamma'',\gamma'}^{-1}$ means we can pass these two copies of G through each other to obtain two entirely disjoint copies of G with the same invariant, and thus the invariant of the doubled graph is the product of the invariant of the two labeled graphs separately. On the other hand, since the transformation of Figure 4 does not change the invariant, it equals the invariant of one copy of the embedded G with edges labeled by γ_i and vertices labeled by $I(x_{j,\alpha_j}' \otimes x_{j,\alpha_j}'')$. Summing over all β we get the invariant of G labeled by l.

Now $\mathfrak{Z}(\Sigma_g)$ is the space spanned by all nonzero labelings of a fixed graph embedded in a handlebody H_g with boundary Σ_g whose image is a retract of the handlebody. The map I defined above then gives a linear map $\Phi \colon \mathfrak{Z}(\Sigma_g) \to \mathfrak{Z}'(\Sigma_g) \otimes \mathfrak{Z}''(\Sigma_g)$ which is easily seen to be an isomorphism (the map I above provides the inverse).

If M is a 2-framed manifold presented by an embedding of $\bigcup_{i=1}^n H_{g_i}$ into S^3 and a surgery link on the complement, then the value of the functional on a vector given by labelings of graphs in H_{g_i} is (up to some normalizations) the invariant applied to the embeddings of these graphs together with the surgery link, with each component labeled by ω. Now notice that since $\Gamma = \Gamma' \times \Gamma''$ we have $\omega = \omega' \otimes \omega'' = \Phi(\omega' \otimes \omega'')$. Thus the image under Φ of the tensor product of vectors coming from labelings which compute $\mathfrak{Z}'(M)(v)$ and $\mathfrak{Z}''(M)(w)$ is the vector coming from the labeling which computes $\mathfrak{Z}(M)(\Phi(v \otimes w))$, so

$$\mathfrak{Z}(M) \circ \Phi = \mathfrak{Z}'(M) \otimes \mathfrak{Z}''(M)$$

after the appropriate identification of the domain spaces. $\square$
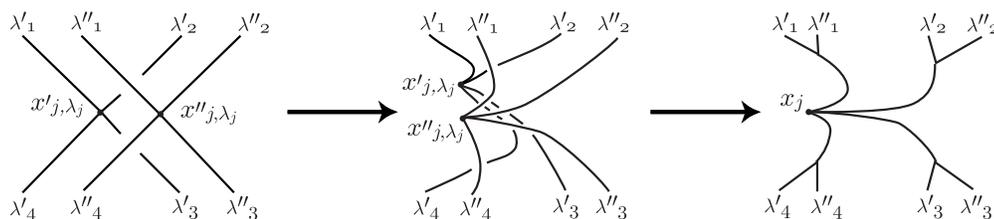
FIGURE 4. Pictorial version of the I isomorphism

REFERENCES

[1] H. H. Andersen and J. Paradowski. Fusion categories arising from semisimple Lie algebras. *Commun. Math. Phys.*, 169(3):563–588, 1995.

[2] S. Axelrod and I. M. Singer. Chern-Simons perturbation theory. In S. Catto and A. Rocha, editors, *Proc. XXth DGM Conference*, pages 3–45. World Scientific, Teaneck, 1992.

[3] S. Axelrod and I. M. Singer. Chern-Simons perturbation theory, II. *J. Differential Geom.*, 39(1):173–213, 1994.

[4] D. Bar-Natan. *Perturbative Aspects of the Chern-Simons Topological Quantum Field Theory*. PhD thesis, Princeton University, 1991.

[5] A. Bruguières. Catégories prémodulaires, modularisations et invariants des variétés de dimension 3. preprint, 1999.

[6] R. Dijkgraaf and E. Witten. Topological gauge theories and group cohomology. *Commun. Math. Phys.*, 129:393–429, 1990.

[7] G. Felder, K. Gawedski, and K. Kupianen. Spectra of Wess-Zumino-Witten models with arbitrary simple groups. *Commun. Math. Phys.*, 117:127–158, 1988.

[8] C. Frohman and J. Kania-Bartoszyńska. so(3)-topological quantum field theory. *Comm. Anal. Geom.*, 4(4):589–679, 1996.

[9] D. Gepner and E. Witten. String theory on group manifolds. *Nuclear Phys. B*, 278(3):493–549, 1986.

[10] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer-Verlag, New York, 1972.

[11] V. F. R. Jones. A polynomial invariant for knots via Von Neumann algebras. *Bull. Amer. Math. Soc.*, 12(1):103–111, 1985.

[12] V. G. Kac and D. Petersen. Infinite dimensional Lie algebras, theta functions and modular forms. *Adv. in Math.*, 53(2):125–264, 1984.

[13] C. Kassel. *Quantum Groups*, volume 155 of *GTM*. Springer-Verlag, New York, 1995.

[14] R. Kirby and P. Melvin. The 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2,\mathbf{c})$. *Invent. Math.*, 105(3):473–545, 1991.

[15] A. A. Kirillov. On an inner product in modular tensor categories. *J. Amer. Math. Soc.*, 9(4):1135–1168, 1996.

[16] M. Kontsevich. Feynman diagrams and low-dimensional topology. In *First European Congress of Mathematics, II (Paris, 1992)*, volume 120 of *Prog. Math.*, pages 97–124. Birkäuser, Basel, 1994.

[17] S. Mac Lane. *Categories For the Working Mathematician*. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg-Berlin, 1971.

[18] G. Moore and N. Seiberg. Taming the conformal zoo. *Phys. Lett. B*, 220(3):422–430, 1989.

[19] M. Müger. Galois theory for braided tensor categories and the modular closure. preprint, 1999.

[20] H. Murakami, T. Ohtsuki, and M. Okada. Invariants of three-manifolds derived from linking matrices of framed links. *Osaka J. Math.*, 29:545–572, 1992.

[21] N. Yu Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127:1–26, 1990.

[22] N. Yu Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103:547–597, 1991.

[23] L. Rozansky. The trivial connection contribution to Witten's invariant and finite-type invariants of rational homology spheres. *Comm. Math. Phys.*, 183(1):23–54, 1997.

[24] S. Sawin. Closed subsets of the Weyl alcove and TQFTs. preprint.

[25] S. Sawin. Three-dimensional 2-framed TQFTs and surgery. preprint.

[26] S. Sawin. Links, quantum groups, and TQFTs. *Bull. Amer. Math. Soc.*, 33(4):413–445, 1996.

[27] V. G. Turaev. *Quantum Invariants of Knots and 3-Manifolds*. Studies in Mathematics. Walter de Gruyter and Co., Berlin, 1994. ISBN 3-11-013704-6.

[28] E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys*, 121(3):351–399, 1989.

FAIRFIELD UNIVERSITY, (203)254-4000x2573, SSAWIN@FAIR1.FAIRFIELD.EDU