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ON FINITENESS OF CERTAIN VASSILIEV INVARIANTS

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**Abstract**

The best known examples of Vassiliev invariants are the coefficients of a Jones-type polynomial expanded after exponential substitution. We show that for a given knot, the first  $N$  Vassiliev invariants in this family determine the rest for some integer  $N$ .

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In this paper we prove a finiteness property for certain families of Vassiliev knot invariants. Let  $V_K(t)$  denote the Jones polynomial [4] of a classical knot or a link  $K$  (i.e. smoothly embedded circle(s) in the 3-space). It is known [2, 1] that the coefficients of powers of  $x$  in  $V_K(e^x)$  are Vassiliev invariants of finite type. More specifically, if we write  $V_K(e^x) = \sum_{n=1}^{\infty} v_n(K)x^n$ , then  $v_n(K)$  is a Vassiliev invariant of type  $n$ . Let us call  $\{v_n(K)\}$  the *Jones-Vassiliev* invariants.

More generally, if  $P_K(\alpha, z)$  denotes the HOMFLY polynomial [3, 4] and  $Y_K(\alpha, z)$  the Kauffman polynomial [5], we obtain a Laurent polynomial (possibly times  $t^{1/2}$ ) by substituting  $t^{1/2} - t^{-1/2}$  for  $z$  and  $t^a$  for  $\alpha$ , where  $a$  is a nonnegative half-integer. Again substituting  $e^x$  for  $t$ , the  $n$ th coefficient of the power series in  $x$  is a degree  $n$  Vassiliev invariant [2, 1]. Let us call these Vassiliev invariants respectively  $p_{a,n}(K)$  and  $f_{a,n}(K)$ . We prove the following.

**Theorem 1** *Given a projection of a link  $K$ ,  $v_n(K)$  is a linear combination of the quantities  $v_0(K), \dots, v_N(K)$ , where  $N$  is the number of crossings in the projection. The coefficients of the linear combination depend on the following data:  $n$ ,  $N$ , the number of separable pieces in  $K$ , the writhe of the projection, and the number of components of the link obtained by parallel smoothings at positive crossings and transverse smoothings at negative crossings that are defined in [10].*

**Theorem 2** *Given a projection of a link  $K$  and a half-integer  $a$ ,  $p_{a,n}(K)$  is a linear combination of  $p_{a,0}(K), \dots, p_{a,N}(K)$ , where  $N$  is  $(2a - 1)(s - 1) + c$ ,  $s$  is the number of Seifert circles in the diagram, and  $c$  is the crossing number of the link. The coefficients of the linear combination depend only on  $n$ ,  $N$ , and the writhe of the diagram.*

**Theorem 3** *Given a projection of a link  $K$  and a half-integer  $a$ ,  $f_{a,n}(K)$  is a linear combination of  $f_{a,0}(K), \dots, f_{a,N}(K)$  where  $N$  is the greatest integer less than  $3ac$ , for  $c$  the crossing number, and the linear combination depend only on  $n$ ,  $N$ , the number of positive and the number of negative crossings in the diagram, and the number of connected components of the diagram.*

These theorems follow from bounds on the degrees of the link polynomials together with the following straightforward lemma.

**Lemma 1** *If  $F(t)$  is of the form  $t^M$  times a degree  $N$  polynomial  $P(t)$  in  $t$ , so that  $F(t) = t^M P(t)$ , for some  $M \in \mathbf{R}$ , and the largest and smallest powers of  $t$  in  $F$  are  $L$  and  $M$  respectively (so that  $N = L - M$ ), then there exist linear functions  $f_{L,N,j}(x_0, \dots, x_N)$ , for each positive integer  $j$ , such that*

$$d^j/dx^j F(e^x)|_{x=0} = f_{L,N,j}(F(1), d/dx(F(e^x))|_{x=0}, \dots, d^N/dx^N(F(e^x))|_{x=0}).$$

*Proof.* Consider the linear ODE

$$D^{N+1}e^{-Mx}f(x) = 0,$$

where  $D = d/d(e^x) = e^{-x}d/dx$ . Notice that changing variables via  $t = e^x$  this is just  $(d^{N+1}/dt^{N+1})t^{-M}f(\ln t) = 0$ , so in particular  $f(x) = F(e^x)$  is a solution. Since this is a linear  $N + 1$ st order differential equation in  $x$ , any solution is a linear combination of the  $N + 1$  fundamental solutions with the coefficients being certain linear combinations of the initial data. By the initial data we mean the first  $N$  derivatives of the solution evaluated at  $x = 0$ , which in this case are  $F(1), d/dx(F(e^x))|_0, \dots, d^{L-M}/dx^{L-M}(F(e^x))|_0$ . By differentiating this linear combination  $j$  times, we see that  $(d^j/dx^j)F(e^x)|_0$  is a linear combination of the initial data too.  $\square$

*Proof (of Theorem 1).* By [6, 10, 11] the span of the Jones polynomial (the highest degree minus the lowest degree) is bounded by  $c + g - 1$ , where  $c$  is the crossing number and  $g$  is the number of disconnected components of the projection. Further, the upper and lower bounds depend on the quantities given in the statement of the theorem. But notice that  $V_K(t)$  is divisible by  $(t^{1/2} + t^{-1/2})^{g-1}$ , since this is true of a link with  $g$  or more unlinked unknots, and  $V_K(t)$  is a linear combination of such by the skein relations. The result of dividing  $V_K(t)$  by this factor is a Laurent polynomial (possibly times  $t^{1/2}$ ) with span bounded by exactly  $c$ . By the lemma, the  $j$ th derivative of this Laurent polynomial evaluated at  $t = 1$  can be written explicitly as a linear combination of the various  $k$ th derivatives for  $0 \leq k \leq c$ . Since the  $j$ th derivative of  $V_K(t)$  is a linear combination of the  $k$ th derivative of this polynomial for  $k \leq j$ , and vice versa, this gives the result.  $\square$

**Corollary 1** *If a knot  $K$  has a projection with  $N$  crossings, and  $v_k(K) = 0$  for  $1 \leq k \leq N$ , then the Jones polynomial of  $K$  is 1.  $\square$*

*Proof (of Theorem 2).* It was proved in [9] that

$$d_{\max}(z) \leq c - s + 1$$

and

$$w - s + 1 \leq d_{\min}(\alpha) \leq d_{\max}(\alpha) \leq w + s - 1$$

where  $d_{\min}$  and  $d_{\max}$  denote the lowest and highest degrees of the respective variable in  $P_K(\alpha, z)$ , and  $c$ ,  $s$  and  $w$  are respectively the crossing number, number of Seifert circles and writhe of the projection of a knot  $K$ .

Notice here that the negative powers of  $z$  arise from the *loop value* (the contribution of a disjoint trivial circle in the skein computations, see [5])  $\delta = (t^a - t^{-a})/(t^{1/2} - t^{-1/2})$  which is a power of  $t$  times a polynomial in  $t$ . Thus terms with negative powers of  $z$  have lower  $d_{\max}(t)$  and higher  $d_{\min}(t)$  than the degree of  $\alpha$  would indicate. So the largest power of  $t$  which can occur would come from the product of the largest power of  $\alpha$  with the largest power of  $z$ , and the smallest power of  $t$  comes from the smallest power of  $\alpha$  times the largest power of  $z$ . Since the largest power of  $z$  occurs multiplied by the lowest and highest power of  $\alpha$ , we have after substitution that

$$a \cdot d_{\min}(\alpha) - d_{\max}(z)/2 \leq d_{\min}(t) \leq d_{\max}(t) \leq a \cdot d_{\max}(\alpha) + d_{\max}(z)/2$$

and hence we get the highest and lowest degrees bounded by  $aw \pm N/2$ . From this and the lemma the result follows.  $\square$

*Proof (of Theorem 3).* The skein definition of the Kauffman polynomial (Dubrovnik version) is given as follows. Let  $D$  be the polynomial defined by the skein relation

$$\begin{aligned} D_{K_+} - D_{K_-} &= z(D_{K=} - D_{K_{\setminus c}}) \\ D_{K_{<+>}} &= \alpha D_K \\ D_{K_{<->}} &= \alpha^{-1} D_K \end{aligned}$$

where  $K_+$ ,  $K_-$  are link diagrams with positive and negative crossings at a single particular crossing point respectively (and the rest of their diagrams coincide),  $K_=\text{ and }K_{\setminus}$  are link diagrams obtained by two ways of smoothings at the crossing. In the second and third equalities  $K_{\langle + \rangle}$  (resp.  $K_{\langle - \rangle}$ ) denotes the link diagram  $K$  with a small positive (resp. negative) kink added.

Now the Kauffman polynomial [7] is defined by

$$Y_K(\alpha, z) = \alpha^{-w(K)} D(\alpha, z)$$

where  $w(K)$  is the writhe of the diagram  $K$ .

Note that the loop value is  $\mu = z^{-1}(\alpha - \alpha^{-1}) + 1$ .

As in the case of HOMFLY, the negative terms of  $z$  come from the loop value, and do not contribute to our estimates of the degrees in  $t$ .

Now we estimate the degrees with respect to  $\alpha$ ,  $\mu$ , and  $z$ . Each branch of the skein tree of  $D_K$  (corresponding to a state  $\sigma$ , which is a choice of smoothings of some of the crossings that results in a collection of framed unlinked unknots) will contribute (a linear combination of) some terms of  $\alpha^p \mu^q z^r$ . Some of them may cancel out each other but we only need an estimate of bounds of the degrees so that we pick the highest and lowest possible degrees among them.

The degree of  $z$  is the number of smoothings we performed in the skein tree. The degree of  $\mu$  is the number of components of the state, and the degree of  $\alpha$  is the the writhe of the state. Thus we obtain the following estimates:

$$\begin{aligned} d_{\max}(z) &\leq \max\{n(\sigma)\} &&\leq c \\ d_{\max}(\alpha) &\leq \max\{w(\sigma)\} &&\leq c_+ \\ d_{\min}(\alpha) &\geq \min\{w(\sigma)\} &&\geq c_- \\ d_{\max}(\mu) &\leq \max\{\ell(\sigma)\} - 1 &&\leq c + g - 1 \end{aligned}$$

where  $w(\sigma)$  and  $\ell(\sigma)$  are the writhe and number of components of  $\sigma$  respectively,  $n(\sigma)$  is the number of crossings smoothed to get to  $\sigma$ , and  $c_+$ ,  $c_-$ , and  $g$  are the number of positive crossings, the number of negative crossings, and the number of connected components of the diagram of  $K$  respectively.

Thus after substitution  $z = t^{1/2} - t^{-1/2}$  and  $\alpha = t^a$ , we get

$$\begin{aligned} d_{\max}(t) &\leq a \cdot d_{\max}(\alpha) + d_{\max}(z)/2 + (a - 1/2)d_{\max}(\mu) \\ &\leq ac_+ + c/2 + (a - 1/2)(c + g - 1) \end{aligned}$$

$$\begin{aligned} d_{\min}(t) &\geq a \cdot d_{\min}(\alpha) - d_{\max}(z)/2 - (a - 1/2)d_{\max}(\mu) \\ &\geq ac_- - c/2 - (a - 1/2)(c + g - 1). \end{aligned}$$

The result then follows from the Lemma. This gives a bound of

$$N \leq 3ac + (2a - 1)(g - 1),$$

but the same argument as in the proof of Theorem 1 gives

$$N \leq 3ac. \square$$

**Remark 1** It has been conjectured that the Jones polynomial detects knotting, *i.e.*, every nontrivial knot has a nontrivial Jones polynomial. By the above Corollary this can be restated as: If a knot is nontrivial, then it has a nontrivial Jones-Vassiliev invariant among those up to  $N$ , the number of crossings of the given knot diagram. Moreover, we can consider the question whether there exist knots whose knottedness is undetectable by *any* Vassiliev invariants of small order in relation to the size of the knot diagram. Here “small” means an order less than or equal to the number of crossings in a minimal diagram representing the knot, but it may be that one wants to adjust the concept of small for this more general problem. It may be useful to consider this generalization of the problem of knot detection in the light of our work.

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