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ABSTRACT. We develop the basic representation theory of all quantum groups at all roots of unity, including Harish-Chandra's Theorem, which allows us to show that an appropriate quotient of a subcategory gives a semisimple ribbon category. This work generalizes previous work on the foundations of representation theory of quantum groups at roots of unity which applied only to quantizations of the simplest groups, or to certain fractional levels, or only to the projective form of the group. The second half of this paper applies the representation to give a sequence of results crucial to applications in topology. In particular for each compact, simple, simply-connected Lie group we show that at each integer level the quotient categopry is in fact modular (thus leading to a Topological Quantum Field Theory), we determine when at fractional levels the corresponding category is modular, and we give a quantum version of the Racah formula for the decomposition of the tensor product.

CONTENTS

INTRODUCTION

In [Wit89] Witten argued that Chern-Simons theory for a compact, connected, simply-connected simple Lie group at integer level k should yield an invariant of links in a (biframed) three-manifold. He also sketched how to compute this invariant combinatorially using two-dimensional conformal field theory, and worked out the SU(2) invariant in enough detail to demonstrate that if well-defined it would have to give the Jones polynomial. in [RT91], Reshetikhin and Turaev constructed an invariant that met all of Witten's criteria using the quantum group associated to $\mathfrak{sl}_2$ (the complexification of the Lie algebra of SU(2)) with the quantum parameter equal to a root of unity depending on the level k. At this point an overall program was clear. To each simple Lie algebra there was associated a quantum group. By understanding the representation theory of this quantum group at roots of unity in an analogous

fashion to Reshetikhin and Turaev's work on $\mathfrak{sl}_2$, one could presumably show that this representation theory formed a *modular tensor category,* and thus construct an invariant of links and three-manifolds, presumably the one Witten associated to the corresponding compact, simple Lie group.

Remarkably, the intervening twelve years have not sufficed to complete this apparently straightforward program. The difficulty is even more surprising considering that around the same time, an analogy was identified between quantum groups at roots of unity and modular groups, which brought both the attention of many skilled algebraists and a host of useful techniques to bear on questions of the representation theory of quantum groups at roots of unity.

Much of the obstacle to the complete resolution of this problem appears to be faulty communication between those working on the algebraic questions and the topologists and mathematical physicists interested in the link and three-manifold invariants. There are two basic confusions. First, most of the algebraic work has been done at roots of unity of odd degree (in fact, degree prime to the entries of the Cartan matrix), while the values that correspond to integer levels and hence the cases of primary interest for their relation to physics are roots of unity of even degree (in fact degrees which are multiples of the entries of the Cartan matrix). Second, much of the algebraic work deals only with representations with highest weights in the root lattice, while in topology and physics one is interested in all representations whose highest weights lie in the weight lattice (indeed the most straightforward invariants geometrically, those coming from the simply-connected groups, require consideration of all the representations, though most of the information is available from the invariants depending only on root lattice representations [Saw02a, Saw02b]). Thus remarkably, there is no proof in the literature that from any quantum group at roots of unity one can construct three-manifold invariants, or even semisimple ribbon categories, though of course these facts are widely understood to be true.

To explain the first confusion requires a careful understanding of the parameter in the definition of the quantum group. Since this parameter always appears with an exponent which is the inner product of elements of the dual to the Cartan subalgebra, it amounts to a choice of scale for this inner product. We work with a parameter q normalized so that the definition of the quantum group and the basic representation theory require only integral powers of q. Our q corresponds to that of most authors, including Lusztig [Lus88, Lus89, Lus90a, Lus90c, Lus90b, Lus92, Lus93] (who sometimes calls it v), Andersen et al [APW91, AK92, APK92, And92, And93, And94, AJS94, And95, APW95, AP95, And97b, And97a, And00, AP00, And01], Kirillov [Kir96] and Chari and Pressley [CP94]. However, because $\mathfrak{sl}_2$ uniquely among Lie algebras has all entries in its Cartan matrix divisible by 2, quantum $\mathfrak{sl}_2$ can be defined with only integral powers of q^2, and thus it is conventional when considering only $\mathfrak{sl}_2$ to refer to what we call q^2 as q. This is the convention in, for example, Reshetikhin and Turaev [RT90, RT91], Kirby and Melvin [KM91]. Our q^2 is also the t of the Jones polynomial [Jon85] and hence A^4 in Kauffman's bracket [Kau87, Kau90]. In our normalization the level k of Witten's invariant corresponds to quantum groups at level $e^{\pi i/D(k+\check{h})}$ where $\check{h}$ is the dual Coxeter number, and D is the ratio of the square

of the length of a long root to that of a short root (i.e., the biggest absolute value of an off-diagonal entry of the Cartan matrix). In particular it is an lth root of unity where l is a multiple of $2D$.

Further confusing the issue is that when the determinant of the Cartan matrix is not 1, the R-matrix cannot be defined (at least in the presence of the weight lattice representations) without introducing fractional powers of q. We recover integral powers if we express everything in terms of the parameter s, where $q = s^L$, with L the index of the root lattice in the weight lattice. Also, some authors use a slightly different set of generators which gives a more symmetric presentation but requires half-integer powers of q and s, including Rosso, Kirby and Melvin, and Kirillov [Ros88, Ros90, KM91, Kir96].

On the algebraic side, the basic facts of quantum groups at roots of unity are somewhat easier if one assumes that the degree of the root of unity is odd and prime to D. Much of the work in this field has focused on the relationship between the representation theory for algebraic groups over a field of prime characteristic p and the representation of the corresponding quantum group at a pth root of unity. Consequently the restriction on the degree of the root of unity seemed harmless and most early work in the field maintained that restriction [Lus89, APW91, AK92, APK92, And92, And93, And94, AJS94, And95, APW95, CP94, Par94]. Thus the results in these works apply to an entirely disjoint set of situations from the quantum groups that arise in connection with affine Lie algebras, conformal field theory and three-manifold invariants.

In fact, for the elementary sorts of results about the representation theory of quantum groups that are necessary to construct the three-manifold invariants and similar tasks, the restrictions on the degree of the root of unity do not appear to be essential, and in fact Andersen and Paradowski [AP95] reproduce many of these results in the general situation, in particular the quantum version of the Racah formula for the tensor product of representations. However, Andersen and Paradowski restrict attention to representations whose weights lie in the root lattice. This restriction corresponds on the Lie group level to considering only the adjoint form of the group. To understand the geometry of the construction it would be best to be able to construct the invariant for all forms of the group.

This article redresses this lack, proving the fundamental results on the representation of quantum groups needed for applications to topology and physics for all nongeneric values of the parameter and all representations. The proofs rely very little on technology borrowed from algebraic groups over finite fields, and as such one can hope they will be more accessible to researchers interested in quantum invariants. The tools are those parts of classical Lie algebra theory that generalize directly to quantum groups, up to the PBW theorem and Harish-Chandra's theorem, together with the R-matrix and Lusztig's integral form.

Section 1 fixes notation and defines the general quantum group $U_q(\mathfrak{g})$ over $\mathbb{Q}(q)$, reviewing some elementary results from the literature. The key nontrivial results we will use are the quantum version of the Poincaré-Birkhoff-Witt Theorem and the existence of the integral form $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$, both due to Lusztig. Also in the section is the

construction of the integral form $U_{\mathcal{A}'}^\dagger(\mathfrak{g})$, which is literally a ribbon Hopf algebra. The existence of such a form is new, and although the construction is not deep, this form is crucial at several points in the article, and seems interesting in its own right.

Section 2 constructs the quantum group at roots of unity (two forms, the standard $U_{\mathbf{q}}^{\mathrm{res}}(\mathfrak{g})$ and the ribbon Hopf algebra $U_{\mathbf{s}}^\dagger(\mathfrak{g})$). It also defines the affine Weyl group and uses it to prove Theorem 1, the quantum version of Harish-Chandra's Theorem (that is, that Weyl modules have the same character if and only if they are related by the affine Weyl group. We do not characterize the center of the quantum group, which is also often referred to as Harish-Chandra's Theorem). The description of the affine Weyl group, which is entirely elementary, has one point worthy of comment, though it is not new. The action of the affine Weyl group depends subtly on the divisibility of the degree of the root of unity, as was observed in [AP95]. At roots of unity corresponding to integer levels it is what is traditionally called the affine Weyl group of $\mathfrak{g}$ in Lie algebra theory and loop groups, while at certain fractional levels it is the affine Weyl group of the dual Lie algebra. The existence of $(2+1)$-dimensional theories at fractional level with a nonstandard Weyl alcove has not been widely recognized and does not seem to have a counterpart in the physics literature.

Harish-Chandra's Theorem is proven for $U_{\mathbf{s}}^\dagger(\mathfrak{g})$. It is not in general true for $U_{\mathbf{q}}^{\mathrm{res}}(\mathfrak{g})$, though a key consequence, the Linkage Principle, is. The proof follows the classical proof, except that it relies explicitly on the R-matrix, and at fractional levels supplements the Harish-Chandra map to the center (which is not onto) with additional central elements not in the range. The same argument gives Harish-Chandra's theorem (with the ordinary Weyl group) for $U_{\mathcal{A}'}^\dagger$.

Section 3 (working over $U_{\mathbf{q}}^{\mathrm{res}}(\mathfrak{g})$ and $U_{\mathbf{s}}^\dagger(\mathfrak{g})$), defines tilting modules as in [Par94, AP95], proves they form a tensor category, and shows that each is a sum of highest weight tilting modules. This section expands and simplifies the work in these two references, using only the elementary theory of quantum groups themselves, rather than techniques from algebraic groups.

Section 4 identitifes the so-called negligible modules, which have the property that all intertwiners from the module to itself have quantum trace zero. This implies in particular that the link invariant is trivial whenever a component is labeled by a negligible module. It is shown here that in fact every highest weight tilting module outside the Weyl alcove is negligible. This section follows the arguments in [Par94, AP95] closely.

Section 5 gives an application of the technology developed in the previous sections. Specifically it gives a quantum version of the Racah formula (also called the Racah-Speiser formula, see MacFarlane et al.[MOR67]) which expresses multiplicities of the tensor products of two Weyl modules in terms of weight multiplicities. A version of this formula for Weyl modules whose highest weights are required to be in the root lattice appears in [AP95].

Finally Section 6 addresses the link and three-manifold invariants and TQFTs constructable from $U_{\mathbf{s}}^\dagger$. The ribbon category associated to the set of all modules and the one associated to the set of all tilting modules are given. The quotient

of the tilting modules by the negligible tilting modules is given and proven to be a semisimple ribbon category. In the physically interesting case, namely where $2D$ divides l, the resulting category is shown to be modular, using the previous sections and well-known results. The behavior at other roots of unity is analyzed and the cases when the theory is modular and when it admits a TQFT or Spin TQFT are identified. These results are all new, although the results on the physically interesting case have been widely understood to be true for years.

The details of many of the following constructions will depend heavily on several parameters depending on the root system, including the ratio D of the square lengths of the long and short roots, the smallest integer L such that L times any inner product of weights is an integer (L always divides the index of the fundamental group, that is of the weight lattice modulo the root lattice), the Coxeter number h and the dual Coxeter number $\check{h}$. For convenience we summarize these quantities (all information taken from [Hum72]).

	A_n	B_{2n+1}	B_{2n}	C_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8	F_4	G_2
L	$n+1$	2	1	1	2	4	3	2	1	1	1
D	1	2	2	2	1	1	1	1	1	2	3
h	$n+1$	$4n+2$	$4n$	$2n$	$4n-2$	$4n$	12	18	30	12	6
$\check{h}$	$n+1$	$4n+1$	$4n-1$	$n+1$	$4n-2$	$4n$	12	18	30	9	4

1. Quantum Groups

We follow Humphreys [Hum72] for all our results on Lie algebras, and for the most part, notation. The following paragraph follows [Hum72] Chapter 10.

Let $\mathfrak{g}$ be a complex simple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra and $\mathfrak{h}^*$ its dual vector space. Let $\Phi \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$. Let $\langle \cdot, \cdot \rangle$ be the unique inner product on $\mathfrak{h}^*$ (and hence on $\mathfrak{h}$) such that $\langle \alpha, \alpha \rangle = 2$ for every short root $\alpha \in \Phi$ (this convention guarantees that the inner product of two roots is an integer. It differs from $(\cdot, \cdot)$ normalized in the convention of physics and loop groups, so that the longest root has length 2. $(\cdot, \cdot) = \langle \cdot, \cdot \rangle / D$, where $D = 3$ for G_2, $D = 2$ for B_2, C_n, and F_4 and $D = 1$ otherwise.). Let $\check{\Phi} = \{\check{\alpha} = 2\alpha/\langle \alpha, \alpha \rangle \,|\, \alpha \in \Phi\}$ be the dual root system to Φ. Let $\Lambda = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}, \, \forall \alpha \in \Phi\}$ be the weight lattice, $\Lambda_r = \mathbb{Z}\Phi \subset \Lambda$ be the root lattice, and $\check{\Lambda}_r = \mathbb{Z}\check{\Phi} \subset \frac{1}{D}\Lambda$ be the dual root lattice. Let $\mathcal{W}$, the *Weyl group*, be the group of isometries of $\mathfrak{h}^*$ generated by reflections about hyperplanes perpendicular to the roots $\alpha \in \Phi$. Thus in particular for each root $\alpha \in \Phi$ we have a reflection $\sigma_\alpha \in \mathcal{W}$ defined by $\sigma_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$. In fact we will most often be interested in the translated action of the Weyl group, which is defined by $\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho$, where $\rho = \sum_{\alpha > 0} \alpha/2$.

Let L be the least integer such that $L\langle \lambda, \gamma \rangle \in \mathbb{Z}$ whenever $\lambda, \gamma \in \Lambda$. Let $\Delta = \{\alpha_1, \ldots, \alpha_N\} \subset \Phi$ be a base, $(a_{ij}) = \langle \alpha_i, \check{\alpha}_j \rangle$ be the Cartan matrix, and let $\alpha > \beta$ mean $\alpha - \beta$ is a nonnegative linear combination of the elements of Δ. Let $\Lambda^+ = \{\lambda \in$

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$\Lambda | \langle \lambda, \alpha_i \rangle \geq 0, \forall \alpha_i \in \Delta \}$ be the set of nonnegative integral weights. Let θ be the longest root, i.e., the unique long root in $\Phi \cap \Lambda^+$, and let ϕ be the unique short root in the same intersection. Finally, let

$\check{h} = \langle \rho, \check{\theta} \rangle + 1$ be the *dual Coxeter number* and $h = \langle \rho, \check{\phi} \rangle + 1$ be the *Coxeter number*.

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. Given integers m, n let

$$[n]_q = (q^n - q^{-n})/(q - q^{-1}) \in \mathcal{A},$$

$$[n]_q! = [n]_q \cdot [n-1]_q \cdots [1]_q \in \mathcal{A},$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = [m]_q!/([n]_q![m-n]_q!) \in \mathcal{A}.$$

Let $d_i = \langle \alpha_i, \alpha_i \rangle / 2$ (so $d_i = 1$ for short roots and $d_i = D$ for long) and let $q_i = q^{d_i}$. Following [CP94] (Except we use E and F for their $X^{\pm}$) we define the Hopf algebra $U_q(\mathfrak{g})$ over $\mathbb{Q}(q)$ with generators E_i, F_i, K_i (for $1 \leq i \leq N$)

$$K_i K_j = K_j K_i \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_i E_j K_i^{-1} = q^{\langle \alpha_i, \alpha_j \rangle} E_j, \qquad K_i F_j K_i^{-1} = q^{-\langle \alpha_i, \alpha_j \rangle} F_j,$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} (E_i)^{1-a_{ij}-r} E_j (E_i)^r = 0 \qquad \text{if } i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} (F_i)^{1-a_{ij}-r} F_j (F_i)^r = 0 \qquad \text{if } i \neq j,$$

$$\Delta(K_i) = K_i \otimes K_i,$$

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i,$$

$$\epsilon(K_i) = 1, \qquad \epsilon(E_i) = \epsilon(F_i) = 0.$$

Define $E_i^{(l)} = E_i^l/[l]_{q_i}!$ for $l \in \mathbb{N}$, and likewise for F_i. We define the $\mathcal{A}$-subalgebra $U_{\mathcal{A}}^{\text{res}}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ to be generated by the elements $E_i^{(r)}$, $F_i^{(r)}$, $K_i^{\pm 1}$, for $1 \leq i \leq n$ and $r \geq 1$. We summarize from [CP94][9.3, 10.1] some important facts about $U_q(\mathfrak{g})$, $U_{\mathcal{A}}^{\text{res}}(\mathfrak{g})$, and their respective representation theories.

$U_{\mathcal{A}}^{\text{res}}(\mathfrak{g})$ is an integral form of $U_q(\mathfrak{g})$ in the sense that $U_q(\mathfrak{g}) = U_{\mathcal{A}}^{\text{res}}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{Q}(q)$, and $U_{\mathcal{A}}^{\text{res}}(\mathfrak{g})$ is a free $\mathcal{A}$-algebra. There exist $E_{\beta_1}, F_{\beta_1}, E_{\beta_2}, F_{\beta_2}, \ldots, E_{\beta_N}, F_{\beta_N}, \in U_{\mathcal{A}}^{\text{res}}(\mathfrak{g})$, where $\beta_1, \ldots, \beta_N$ is an enumeration of the positive roots, such that each E_{β_i}, F_{β_i} satisfies $K_j E_{\beta_i} K_j^{-1} = q^{\langle \alpha_j, \beta_i \rangle} E_{\beta_i}$ and the set of all $(E_{\beta_N})^{(l_N)} \cdots (E_{\beta_1})^{(l_1)}$ forms a basis for the subalgebra $U_q^+(\mathfrak{g})$ generated by $\{1, E_i^{(k)}\}$ (and likewise for F, with $U_q^-(\mathfrak{g})$ defined correspondingly).

For each $\lambda \in \Lambda^+$ there is a unique (up to isomorphism) irreducible, $U_q(\mathfrak{g})$-module with a vector v such that $K_i v = q^{\langle \alpha_i, \lambda \rangle} v$ and no vector w satisfies $K_i w = q^{\langle \alpha_i, \gamma \rangle} w$ for $\gamma > \lambda$. This module, called the Weyl module W_q^λ, is a direct sum of its weight

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spaces and the dimensions of these weight spaces are the same as that of the classical Weyl module W^λ. The tensor product of two Weyl modules is isomorphic to a direct sum of Weyl modules with multiplicities the same as those in the classical case. If v is a highest weight vector of a module W^λ (that is a vector of weight λ) then $W_\mathcal{A}^\lambda = U_\mathcal{A}^{\mathrm{res}}(\mathfrak{g}) \cdot v$ is a $U_\mathcal{A}^{\mathrm{res}}(\mathfrak{g})$-submodule of W^λ, an $\mathcal{A}$-form of W^λ, and a direct sum of its intersections with the weight spaces of W^λ, each of which is a free $\mathcal{A}$-module of finite rank.

One of the most important reasons for considering quantum groups is that they are ribbon Hopf algebras, and give invariants of links. It is thus a source of some embarrassment that none of the versions of the quantum group defined here are ribbon Hopf algebras! There are three obstacles to defining a ribbon structure (specifically, to giving an R-matrix). The first was discussed in the introduction, that to define the R-matrix, requires fractional powers of q (Though in the end the properly normalized link invariant contains only integer powers of q, see Le [Le00]). The second is that an insufficient subset of the enveloping algebra of the Cartan subalgebra has been included in these definitions to write down the R-matrix, even formally. The third is that the R-matrix involves an infinite sum, though in any particular finite-dimensional representation only finitely many terms are nonzero. Below we give an integral form of the quantum group which is a ribbon Hopf algebra (technically a topological Hopf algebra, in the sense that the comultiplication maps to a completed tensor product).

Let $\mathcal{A}' = \mathbb{Z}[s, s^{-1}]$ and define a monomorphism $\mathcal{A} \to \mathcal{A}'$ by $q \mapsto s^L$ (henceforth we will treat this monomorphism as an inclusion and simply write $q = s^L$). Define the $\mathcal{A}'$ Hopf algebra $U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g}) = U_\mathcal{A}^{\mathrm{res}}(\mathfrak{g}) \otimes_\mathcal{A} \mathcal{A}'$ and the $U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})$-module $W_{\mathcal{A}'}^\lambda = W_\mathcal{A}^\lambda \otimes_\mathcal{A} \mathcal{A}'$.

Now consider the Hopf algebra of functions on the additive group Λ. The collection of all set-theoretic functions from Λ to $\mathcal{A}'$, $\mathrm{Map}(\Lambda, \mathcal{A}')$, is naturally an algebra over $\mathcal{A}'$ with pointwise multiplication. It is a (topological) Hopf algebra when given the comultiplication $\Delta(f)(\mu, \mu') = f(\mu + \mu')$, the counit $\epsilon(f) = f(0)$, and the antipode $S(f)(\mu) = f(-\mu)$ for $f \in \mathrm{Map}(\Lambda, \mathcal{A}')$ and $\mu, \mu' \in \Lambda$. Here

$$\Delta \colon \mathrm{Map}(\Lambda, \mathcal{A}') \to \mathrm{Map}(\Lambda \times \Lambda, \mathcal{A}').$$

The latter space contains the natural embedding of $\mathrm{Map}(\Lambda, \mathcal{A}') \otimes \mathrm{Map}(\Lambda, \mathcal{A}')$ as a dense subspace in the topology of pointwise convergence, and thus may be viewed as the completed tensor product. A topological basis for this Hopf algebra of functions is given by $\{\delta_\lambda\}_{\lambda \in \Lambda}$, where $\delta_\lambda(\gamma) = \delta_{\lambda, \gamma}$. By topological basis we mean the elements are linearly independent and span a dense subspace of $\mathrm{Map}(\Lambda, \mathcal{A}')$ in the topology of pointwise convergence.

Recall any Abelian group with a homomorphism to its dual has an R-matrix associated to the homomorphism in the Hopf algebra of functions. In the case of the homomorphism $\lambda \mapsto s^{L\langle \lambda, \cdot \rangle}$, the R-matrix is $\sum_{\lambda, \gamma} s^{L\langle \lambda, \gamma \rangle} \delta_\lambda \otimes \delta_\gamma$, which once again is an element not of the tensor product of the Hopf algebra with itself but of the completion. Defining $\lambda_i \in \Lambda$ by $\langle \lambda_i, \check{\alpha}_j \rangle = \delta_{i,j}$ we can write the canonical dual element to the pairing as $\sum_i \lambda_i \otimes \check{\alpha}_i$, and then being somewhat abusive of notation can refer to the R-matrix above as $q^{\sum_i \lambda_i \otimes \check{\alpha}_i}$. In the same vein we shall use q^λ to refer to

the homomorphism $\sum_{\gamma \in \Lambda} s^{L\langle \lambda, \gamma \rangle} \delta_\gamma$. Let $U_{\mathcal{A}'}^\dagger(\mathfrak{h})$ be $\mathrm{Map}(\Lambda, \mathcal{A}')$ viewed as a topological ribbon Hopf algebra.

$U_{\mathcal{A}'}^\dagger(\mathfrak{h})$ acts on $U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})$ via the Λ-grading of $U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})$. Specifically, define the weight of a monomial in $\{E_i, F_i, K_i\}$ to be the sum of α_i for each factor of E_i and $-\alpha_i$ for each factor of F_i. Then $f \in U_{\mathcal{A}'}^\dagger(\mathfrak{h})$ acts on a monomial X by $f[X] = f(\mathrm{weight}(X))X$ and extends linearly. This action is automorphic in the sense that $\Delta(h[X]) = h^{(1)}[X^{(1)}] \otimes h^{(2)}[X^{(2)}]$ and $h[XY] = h^{(1)}[X]h^{(2)}[Y]$ (here we use Sweedler's notation, writing $\Delta(h)$ as $\sum_\beta h_\beta^{(1)} \otimes h_\beta^{(2)}$ and suppressing the index β). As such we can form the semidirect product Hopf algebra $U_{\mathcal{A}'}^\dagger(\mathfrak{h}) \ltimes U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})$ (also called the smash product, see Montgomery [Mon93]). This Hopf algebra is (densely) generated by $\{E_i, F_i, K_i\} \cup \{\delta_\lambda\}_{\lambda \in \Lambda}$, with the standard quantum group relations together with

$$\delta_\lambda \delta_\gamma = \delta_{\lambda, \gamma} \delta_\lambda, \qquad \sum_{\lambda \in \Lambda} \delta_\lambda = 1$$

$$\delta_\lambda K_i = K_i \delta_\lambda \qquad \delta_\lambda E_i = E_i \delta_{\lambda - \alpha_i} \qquad \delta_\lambda F_i = F_i \delta_{\lambda + \alpha_i}.$$

If $U_{\mathcal{A}'}^\dagger(\mathfrak{h}) \ltimes U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})$ acts on an $\mathcal{A}'$-module V, and $v \in V$, we say v is of weight $\lambda \in \Lambda$ if $K_i v = q^{\langle \lambda, \alpha_i \rangle} v$ and $fv = f(\lambda)v$ for $f \in U_{\mathcal{A}'}^\dagger(\mathfrak{h})$, and we say V is a λ weight space if it consists entirely of weight λ vectors. Let $\mathfrak{W}$ be the direct product of all $U_{\mathcal{A}'}^\dagger(\mathfrak{h}) \ltimes U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})$-modules which are a finite direct sum of $\mathcal{A}'$-free λ weight spaces for $\lambda \in \Lambda$. Of course $U_{\mathcal{A}'}^\dagger(\mathfrak{h}) \ltimes U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})$ acts on $\mathfrak{W}$. The kernel of this action is a two-sided ideal I (clearly I includes at least $K_i - q^{\alpha_i}$), and since the tensor product of two finite direct sums of $\mathcal{A}'$-free λ-spaces for $\lambda \in \Lambda$ is another such, I is a Hopf ideal. Thus the quotient $U_{\mathcal{A}'}^\dagger(\mathfrak{h}) \ltimes U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})/I$ is a Hopf algebra which embeds into $\mathrm{End}(\mathfrak{W})$. The product topology on $\mathfrak{W}$ gives $\mathrm{End}(\mathfrak{W})$ a topology, one in which a sequence converges if and only if it converges on each finite-dimensional submodule. The closure of $U_{\mathcal{A}'}^\dagger(\mathfrak{h}) \ltimes U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})/I$ in this topology is what we call $U_{\mathcal{A}'}^\dagger(\mathfrak{g})$. The product, coproduct, antipode and counit clearly extend to the completion (Δ has range the closure of $U_{\mathcal{A}'}^\dagger(\mathfrak{g}) \otimes U_{\mathcal{A}'}^\dagger(\mathfrak{g})$ in $\mathrm{End}_{\mathcal{A}'}(\mathfrak{W} \otimes \mathfrak{W})$, which we will refer to as $U_{\mathcal{A}'}^\dagger(\mathfrak{g}) \overline{\otimes} U_{\mathcal{A}'}^\dagger(\mathfrak{g})$, the completed tensor product).

The reader might reasonably wonder, after the sequence of rather abstract steps in the construction, whether there is anything left to this algebra at all. In fact $W_{\mathcal{A}'}^\lambda$ ($\lambda \in \Lambda^+$) can be made into a $U_{\mathcal{A}'}^\dagger(\mathfrak{h}) \ltimes U_{\mathcal{A}'}^{\mathrm{res}}(\mathfrak{g})$-module by letting $f \in U_{\mathcal{A}'}^\dagger(\mathfrak{h})$ act on a weight λ vector by multiplication by $f(\lambda)$. $W_{\mathcal{A}'}^\lambda$ is a finite direct sum of free weight spaces, as above, so any pair of elements of the semidirect product that act as different endomorphisms on some $W_{\mathcal{A}'}^\lambda$ represents different elements of $U_{\mathcal{A}'}^\dagger(\mathfrak{g})$.

This extended Hopf algebra $U_{\mathcal{A}'}^\dagger(\mathfrak{g})$ is a ribbon Hopf algebra (see [CP94] for the definition). Specifically notice that our earlier R-matrix

$$q^{\sum_i \breve{\alpha}_i \otimes \lambda_i}$$

is an element of $U_{\mathcal{A}'}^{\dagger}(\mathfrak{g}) \,\overline{\otimes}\, U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$. Therefore so is

$$(1) \quad R = q^{\sum_i \tilde{\alpha}_i \otimes \lambda_i} \sum_{t_1,\dots t_N=1}^{\infty} \prod_{r=1}^{N} q_{\beta_r}^{t_r(t_r+1)/2} (1 - q_{\beta_r}^{-2})^{t_r} [t_r]_{q_{\beta_r}}! E_{\beta_r}^{(t_r)} \otimes F_{\beta_r}^{(t_r)}$$

$$\in U_{\mathcal{A}'}^{\dagger}(\mathfrak{g}) \,\overline{\otimes}\, U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$$

where $q_{\beta_r} = q^{d_i}$ when β_r is the same length as α_i. A fairly standard calculation confirms that R is a quasitriangular element for $U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$. Further, notice that the grouplike element q^{ρ} is a charmed element of the Holf algebra for this R, making $U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$ into a ribbon Hopf algebra. In particular, conjugation by $q^{2\rho}$ is the square of the antipode, so that for any finite-dimensional $U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$ module V, free over $\mathcal{A}'$, the functional

$$\mathrm{qtr}_V \colon U_{\mathcal{A}'}^{\dagger}(\mathfrak{g}) \to \mathcal{A}'$$

$$\mathrm{qtr}_V(x) = \mathrm{tr}_V(q^{2\rho} x)$$

is an invariant functional on $U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$ in the sense that $\mathrm{qtr}(a^{(1)} b S(a^{(2)})) = \epsilon(a) \, \mathrm{qtr}(b)$ (using Sweedler's notation as above).

Define the *quantum dimension*

$$\mathrm{qdim}(V) = \mathrm{qtr}_V(1) = \mathrm{tr}(q^{2\rho})$$

and in particular define $\mathrm{qtr}_\lambda = \mathrm{qtr}_{W_{\mathcal{A}'}^\lambda}$ and $\mathrm{qdim}(\lambda) = \mathrm{qdim}(W_{\mathcal{A}'}^\lambda)$. Notice that since $\mathrm{qtr}_{V \otimes W} = \mathrm{qtr}_V \, \mathrm{qtr}_W$,

$$\mathrm{qdim}(V \otimes W) = \mathrm{qdim}(V) \, \mathrm{qdim}(W).$$

Finally, a simple calculation modeled on the classical Weyl character formula gives

$$(2) \qquad \mathrm{qdim}(\lambda) = \prod_{\beta > 0} \left(q^{\langle \lambda+\rho, \beta \rangle} - q^{-\langle \lambda+\rho, \beta \rangle} \right) / \left(q^{\langle \rho, \beta \rangle} - q^{-\langle \rho, \beta \rangle} \right) \in \mathcal{A}.$$

2. Roots of Unity and the Affine Weyl Group

Now restrict the generic q to a root of unity. Specifically, let l be a positive integer, and consider the homomorphism $\mathcal{A}' \to \mathbb{Q}[\mathbf{s}]$, where $\mathbf{s}$ is an abstract primitive lLth root of unity (i.e. satisfies the lLth cyclotomic polynomial) given by $s \mapsto \mathbf{s}$. As before define $U_{\mathbf{s}}^{\dagger}(\mathfrak{g}) = U_{\mathcal{A}'}^{\dagger}(\mathfrak{g}) \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$, $W_{\mathbf{s}}^\lambda = W_{\mathcal{A}'}^\lambda \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$. Write $\mathbf{q} = \mathbf{s}^L$. Likewise $q \mapsto \mathbf{q}$ gives a homomorphism $\mathcal{A} \to \mathbb{Q}[\mathbf{q}]$, and $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g}) \to U_{\mathbf{q}}^{\mathrm{res}}(\mathfrak{g})$. Write $\mathbf{q}^\lambda$ for $q^\lambda \otimes 1 \in U_{\mathcal{A}'}^{\dagger}(\mathfrak{g}) \otimes \mathbb{Q}[\mathbf{q}] = U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$.

Notice $U_{\mathbf{s}}^{\dagger} \otimes U_{\mathbf{s}}^{\dagger} \cong (U_{\mathcal{A}'}^{\dagger} \otimes U_{\mathcal{A}'}^{\dagger}) \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$ embeds naturally (and densely in the inherited topology) into $(U_{\mathcal{A}'}^{\dagger} \,\overline{\otimes}\, U_{\mathcal{A}'}^{\dagger}) \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$, and thus we may define the latter space as the completed tensor product $U_{\mathbf{s}}^{\dagger} \,\overline{\otimes}\, U_{\mathbf{s}}^{\dagger}$. $U_{\mathbf{s}}^{\dagger}$ then becomes a Hopf algebra, and in fact a ribbon Hopf algebra since the image of R is in $U_{\mathbf{s}}^{\dagger} \,\overline{\otimes}\, U_{\mathbf{s}}^{\dagger}$.

For each $i \le n$ let l_i be $l/\gcd(l, d_i)$ (that is, the degree of q_i) and let l_i' be l_i or $l_i/2$ according to whether l_i is odd or even (so that l_i' is the least natural number such that $q_i^{l_i'} \in \{\pm 1\}$. Likewise let l' be l or $l/2$ according to whether l is odd or even. Define

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the *affine Weyl group*, $\mathcal{W}_l$, to be the group of isometries of $\mathfrak{h}^*$ generated by reflection about the hyperplanes
$$\langle x, \alpha_i \rangle = \langle k l'_i \alpha_i / 2, \alpha_i \rangle = k l'_i d_i$$
for each $k \in \mathbb{Z}$ and each $\alpha_i \in \Delta$. This includes the Weyl group $\mathcal{W}$ as a subgroup (when $k = 0$). Again we will usually be interested in the translated action of the affine Weyl group, given by $\sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho$.

Lemma 1. *The affine Weyl group is the semidirect product of the ordinary Weyl with the group of translations $l'\check{\Lambda}_r$ if l' is divisible by D or $l'\Lambda_r$ if l' is not divisible by D. In particular a set of generators consists of reflections σ_{α_i}, $\alpha_i \in \Delta$ together with translation by $l'\theta/D$ (if $D|l'$), or $l'\phi$ (if $D \nmid l'$). A fundamental domain for the translated action of the affine Weyl group is the* principal Weyl alcove, C_l, *which is the region $\langle x + \rho, \alpha_i \rangle \geq 0$, $\langle x + \rho, \theta \rangle \leq l'$ (if $D|l'$) or $\langle x + \rho, \alpha_i \rangle \geq 0$, $\langle x + \rho, \phi \rangle \leq l'$ otherwise.*

Proof: Reflection about the hyperplane $\langle x, \alpha_i \rangle = 0$ followed by reflection about $\langle x, \alpha_i \rangle = \langle l'_i \alpha_i / 2, \alpha_i \rangle$ gives translation by $l'_i \alpha_i$, which since $d_i = 1$ or $d_i = D$ and D is prime is translation by $l'\alpha_i$ or $l'\check{\alpha}_i$ according to the divisibility of l'. Conjugation by σ_β for various β gives translation by $l'\gamma$ or $l'\check{\gamma}$ for γ any root. Thus $\mathcal{W}_l$ contains the groups mentioned, and clearly is generated by them. Since the Weyl group acts by conjugation on the group of translations, the full group is a semidirect product.

The subgroup of translations is generated by $l'\check{\beta}$ (resp. $l'\beta$) for β a long root of Φ (resp. β a short root of Φ). Thus a fundamental domain for the group of translations would be the polygon bounded by the hyperplanes $\langle x + \rho, \check{\beta} \rangle \leq l'$ (resp. $\langle x + \rho, \beta \rangle \leq l'$) for all such β. Since this region is invariant under the translated action of $\mathcal{W}$, a fundamental region for $\mathcal{W}_l$ is given by the intersection of this region with a fundamental region of this action of $\mathcal{W}$, which is exactly the region given. $\square$

Remark 1. *When $\mathfrak{g}$ is not simply-laced the affine Weyl group's action is distinctly different if l' is divisible by D versus if it is not. When l' is divisible by D, we recognize the action of the affine Weyl group described by Kac [Kac83] and many other authors discussing affine Lie algebras and loop groups, except that the translations are multiplied by l'/D. This is the affine Weyl group relevant to affine Lie algebras, and the affine Weyl group of the root system Φ as discussed in, for example, Bourbaki [Bou02]. When l' is not divisible by D we recover the affine Weyl group discussed in Jantzen [Jan87] and many other authors considering modular groups. It is (with multiplication by l') the usual affine Weyl group of the dual root system $\check{\Phi}$. It is remarkable that both versions of the affine Weyl group appear, on equal footing, in the context of quantum groups.*

Lemma 2. *The affine Weyl group above is the largest subgroup of $\mathcal{W}^\dagger \overset{\mathrm{def}}{=} \mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$ which fixes the root lattice Λ_r under the translated action. In particular these are equal when $2D|l$.*

Proof: The subgroup of the group of translations $\frac{l}{2}\check{\Lambda}_r$ which preserves Λ_r is $\frac{l}{2}\check{\Lambda}_r \cap \Lambda_r = l'\check{\Lambda}_r \cap \Lambda_r$ since half a dual root is never a dual root. If $D|l'$, then $l'\check{\Lambda}_r \subset \Lambda_r$. If

D does not divide l' then they are relatively prime, so for short roots $l'\check{\beta} = l'\beta \in \Lambda_r$, but for long roots the smallest multiple in Λ_r is $Dl'\check{\beta} = l'\beta$, so $\frac{l}{2}\check{\Lambda}_r \cap \Lambda_r \subset l'\Lambda_r$. Since the translated action of the ordinary Weyl group preserves the root lattice, the result follows. $\qquad\square$

The remainder of the section is devoted to a proof of the quantum version of Harish-Chandra's Theorem for $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$. Harish-Chandra's theorem for $U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$ (with of course the classical Weyl group) works by a perfectly analogous argument (though simpler) and will be left to the reader.

Consider the action of the center Z of $U_{\mathbf{s}}^{\dagger}$ on a Weyl module of highest weight λ. If v is a vector of weight λ then so is zv for any $z \in Z$, so z must act as multiplication by an element of $\mathbb{Q}[\mathbf{s}]$ on v, say $zv = \chi_\lambda(z)v$. Since every element of the Weyl module is of the form Fv for some $F \in U_{\mathbf{s}}^{\dagger -}$ (the subalgebra of $U_{\mathbf{s}}^{\dagger}$ generated by $\{1, F_i^{(k)}\}$) we have $zFv = Fzv = \chi_\lambda(z)Fv$, so that in fact z acts as multiplication by $\chi_\lambda(z)$ on the entire Weyl module. Thus each λ gives us an algebra homomorphism χ_λ from the center Z to $\mathbb{Q}[\mathbf{s}]$.

Definition 1. *If $\lambda, \gamma \in \Lambda$ say that $\lambda \sim \gamma$ if $\chi_\lambda = \chi_\gamma$.*

Now $\chi_\lambda = \chi_\gamma$ if λ occurs as a highest weight in a highest weight γ module, so $\sim$ includes at least the extension of this inclusion relation to an equivalence relation.

Let us first understand the relation $\sim$ in the case $\mathfrak{g} = \mathfrak{sl}_2$. The Verma module of weight $j\theta$, $j \in \mathbb{Z}^{\geq 0}/2 \cong \Lambda^+$, is a $U_{\mathbf{s}}^{\dagger}$-module spanned by $\{F^{(k)}v \mid k \geq 0\}$, where v is of weight $j\theta$ and $E^{(k)}v = 0$ for all $k > 0$. For this module

$$\begin{bmatrix} 2j + s - r \\ s \end{bmatrix}_{\mathbf{q}} E^{(r)}F^{(s)}v = F^{(r-s)}v$$

if $s \leq r$.

Notice $[r]_{\mathbf{q}} = 0$ if and only if r is a multiple of l'. Thus $F^{(s)}v$ is a highest weight vector when either $s = 2j + 1$ or $s = 2j + 1 - kl'$ and $s < l'$. So $j\theta \sim -(j+1)\theta$ when $j \geq 0$ and $j\theta \sim (kl' - j - 1)\theta$ when $2j < (k+1)l'$. By transitivity $j \sim j'$ whenever $j\theta$ is connected to $j'\theta$ by the affine Weyl group.

Now consider a general $\mathfrak{g}$.

Proposition 1. *If $\lambda, \gamma \in \Lambda$ and there is a $\sigma \in \mathcal{W}_l$ such that $\gamma = \sigma \cdot \lambda$, then $\lambda \sim \gamma$.*

Proof: Let $\lambda \in \Lambda$. Recall that the Verma module of highest weight λ can be constructed as follows. Consider $U_{\mathbf{s}}^{\dagger}$ as a $U_{\mathbf{s}}^{\dagger}$-module under the adjoint action, and quotient it by the left ideal generated by $E_i^{(k)}$ and $K_i - q^{(\lambda, \alpha_i)})$ for all $i \leq N$ and $k \in \mathbb{N}$. It is easy to see that the vector 1 (which above was called v) is a highest weight vector of weight λ. Now for each $i \leq N$ the set $\{E_i^{(k)}, F_i^{(k)}, K_i\}$ generate a subalgebra isomorphic to $U_{\mathbf{s}_i}^{\dagger}(\mathfrak{sl}_2)$ (here $\mathbf{s}_i = \mathbf{s}^{d_i}$) and the vectors $F_i^{(k)}v$ span a $U_{\mathbf{s}_i}^{\dagger}(\mathfrak{sl}_2)$ module isomorphic to the Verma module of weight $\langle \lambda, \check{\alpha}_i \rangle / 2$. Therefore the vectors $F_i^{\langle \lambda, \check{\alpha}_i \rangle + 1}v$ and $F_i^{\langle \lambda, \check{\alpha}_i \rangle + 1 - kl_i'}v$ where $\langle \lambda, \check{\alpha}_i \rangle < (k+1)l_i'$ give highest weight vectors. Thus $\lambda \sim \sigma \cdot \lambda$, for σ a generator of the affine Weyl group. The result follows by the transitivity of the $\sim$ relation. $\quad\square$

The other direction of Harish-Chandra's theorem requires the R-matrix. Writing $R = \sum_j u_j \otimes v_j$ (of course the sum is infinite), define

$$D = \left(\sum_j v_j \otimes u_j \right) \left(\sum_k u_k \otimes v_k \right)$$

in $U_{\mathbf{s}}^\dagger(\mathfrak{g}) \,\overline{\otimes}\, U_{\mathbf{s}}^\dagger(\mathfrak{g})$. In turn write $D = \sum_i x_i \otimes y_i$. Let

$$\Psi \colon (U_{\mathbf{s}}^\dagger(\mathfrak{g}))^* \to U_{\mathbf{s}}^\dagger(\mathfrak{g})$$
$$\Psi(z^*) = \sum_i z^*(y_i) x_i,$$

where $(U_{\mathbf{s}}^\dagger(\mathfrak{g}))^*$ is the direct sum over all $\lambda \in \Lambda^+$ of the set of functionals on $U_{\mathbf{s}}^\dagger(\mathfrak{g})$ which factor through the representation on $W_{\mathbf{s}}^\lambda$. This map is called the Drinfel'd map. We will also be interested in

$$D_\mathfrak{h} = \mathbf{q}^{\sum_i \lambda_i \otimes \breve{\alpha}_i} \mathbf{q}^{\sum_i \breve{\alpha}_i \otimes \lambda_i} = \mathbf{q}^{2 \sum_i \lambda_i \otimes \breve{\alpha}_i}$$

and the associated

$$\Psi_\mathfrak{h} \colon (U_{\mathbf{s}}^\dagger(\mathfrak{g}))^* \to U_{\mathbf{s}}^\dagger(\mathfrak{h})$$
$$\Psi_\mathfrak{h}(z^*) = \sum_i z^*(y_i) x_i$$

writing $D_\mathfrak{h} = \sum_i x_i \otimes y_i$.

By the PBW theorem there is a well-defined map

$$\Theta \colon U_{\mathbf{s}}^\dagger(\mathfrak{g}) \to U_{\mathbf{s}}^\dagger(\mathfrak{h})$$

given by sending all products in the PBW basis which contain factors of E_i or F_i to zero and all other products to themselves. Thus $\chi_\lambda = \lambda \circ \Theta$ on the center Z. What's more, since the only terms in D which do not contain factors of the form E and F are those in $D_\mathfrak{h}$,

(3) $$\Theta \Psi = \Psi_\mathfrak{h}.$$

Recall that the adjoint action of of $U_{\mathbf{s}}^\dagger$ on itself is given by

$$\mathrm{ad}_a(x) = a^{(1)} x S(a^{(2)})$$

where we have used Sweedler's notation which writes $\Delta(a) = a^{(1)} \otimes a^{(2)}$ with understood summation sign and indices. An invariant element of $U_{\mathbf{s}}^\dagger$ is then an x such that $\mathrm{ad}_a(x) = \epsilon(a)x$ for all $a \in U_{\mathbf{s}}^\dagger$. We note [Kup91] argues that for any Hopf algebra the map $(a, b) \mapsto (a^{(1)}, ba^{(2)})$ is one-to-one and onto from $U_{\mathbf{s}}^\dagger \,\overline{\otimes}\, U_{\mathbf{s}}^\dagger$ to itself (his argument was for ordinary tensor product, but is easily adapted to our topological situation). For a given $u \in U_{\mathbf{s}}^\dagger$ choosing (a, b) which is mapped to $(u, 1)$, one argues that for an ad-invariant element z, we have $uz = a^{(1)} z S(a^{(2)}) S(b) = \epsilon(a) z S(b) = z a^{(1)} S(a^{(2)}) S(b) = zu$ for all u. Since the converse is clear we conclude that the invariant elements of $U_{\mathbf{s}}^\dagger$ (indeed of any Hopf algebra) are exactly the elements of the center.

Likewise the coadjoint action on $(U_{\mathbf{s}}^\dagger)^*$ sends z^* to

$$\mathrm{coad}_a(z^*) = z^*(a^{(1)} \cdot S(a^{(2)})).$$

Lemma 3. *The Drinfel'd map Ψ takes invariant functionals to the center of $U_{\mathbf{s}}^\dagger$.*

Proof: For notational convenience write
$$\Delta^2(a) = b^{(1)} \otimes b^{(2)} \otimes b^{(3)}$$
and
$$\Delta^3(a) = c^{(1)} \otimes c^{(2)} \otimes c^{(3)} \otimes c^{(4)}.$$

Notice by the quasitriangularity of R
$$\Delta(a)D = D\Delta(a)$$
for all $a \in U_{\mathbf{s}}^\dagger$. Then if z^* is an invariant functional
$$\epsilon(a)\Psi(z^*) = \sum_i z^*(y_i c^{(2)} S(c^{(3)})) x_i c^{(1)} S(c^{(4)})$$
by the basic relations of a Hopf algebra so
$$= \sum_i z^*(c^{(2)} y_i S(c^{(3)})) c^{(1)} x_i S(c^{(4)})$$
$$= \sum_i \mathrm{ad}_{S^{-1}(b^{(2)})}(z^*)(y_i) b^{(1)} x_i S(b^{(3)})$$
$$= \sum_i \epsilon(S^{-1}(b^{(2)})) z^*(y_i) b^{(1)} x_i S(b^{(3)})$$
$$= \sum_i z^*(y_i) a^{(1)} x_i S(a^{(2)}).$$

Thus $\Psi(z^*)$ is an invariant element of $U_{\mathbf{s}}^\dagger$.

$\square$

Corollary 1. *If $\lambda \sim \gamma$, then $\lambda\Psi_\mathfrak{h} = \gamma\Psi_\mathfrak{h}$ on invariant functionals.*

Proposition 2. *Suppose $\lambda, \gamma \in \frac{1}{2L}\check{\Lambda}_r$. If $\lambda\Psi_\mathfrak{h}$ agrees with $\gamma\Psi_\mathfrak{h}$ on invariant functionals, then λ and γ are in the same orbit of $\mathcal{W}^\dagger$. Further, the set $\{\lambda\Psi_\mathfrak{h}\}$, where λ runs through a choice of representative of each equivalence class in $\frac{1}{2L}\check{\Lambda}_r/\mathcal{W}^\dagger$, is a set of linearly independent functionals on the quantum traces.*

Proof: For each $\nu \in \Lambda^+$ the functional qtr_ν is an invariant functional. By induction on the ordering we can form a linear combination of these qtr_ν to produce an invariant functional which on $U_{\mathbf{s}}^\dagger(\mathfrak{h})$ acts as $\sum_{\sigma \in \mathcal{W}} \sigma(\nu)(\mathbf{q}^{2\rho} \cdot)$ for each $\nu \in \Lambda^+$. Notice that $(\lambda \otimes \mu)(D_\mathfrak{h}) = \mathbf{q}^{2\langle \lambda, \mu \rangle}$ for $\mu \in \Lambda$. Thus
$$\lambda(\Psi_\mathfrak{h}(\sum_{\sigma \in \mathcal{W}} \sigma(\nu)(\mathbf{q}^{2\rho} \cdot))) = (\lambda \otimes \sum_{\sigma \in \mathcal{W}} \sigma(\nu)) \left(D_\mathfrak{h}(1 \otimes \mathbf{q}^{2\rho})\right) = \sum_{\sigma \in \mathcal{W}} \mathbf{q}^{2\langle \lambda + \rho, \sigma(\nu) \rangle}.$$

The set of maps $\{\mathbf{q}^{2\langle \lambda + \rho, \cdot \rangle} : \lambda \in \frac{1}{2L}\check{\Lambda}_r/\frac{l}{2}\check{\Lambda}_r\}$ is a basis for maps from $\Lambda/(lL\Lambda)$ to $\mathbb{Q}[\mathbf{s}]$. $\mathcal{W}$ permutes this basis, so $\sum_{\sigma \in \mathcal{W}} \mathbf{q}^{2\langle \lambda + \rho, \sigma(\cdot) \rangle}$ forms a basis for maps from $(\Lambda/(lL\Lambda))^\mathcal{W}$ to $\mathbb{Q}[\mathbf{s}]$, when λ ranges over representatives of each Weyl orbit in $\mathcal{W}^\dagger$. Thus $\lambda\Psi_\mathfrak{h}$ is unchanged by $\mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$, and any set of orbit representatives is linearly independent.

$\square$

Lemma 4. *If $\lambda \sim \gamma$ then $\lambda - \gamma \in \Lambda_r$.*

Proof: Notice an element $f \in U_{\mathbf{s}}^{\dagger}(\mathfrak{h})$ is in the center of $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$ if and only if $f(\lambda) = f(\lambda + \alpha_i)$ for all $\lambda \in \Lambda$ and all i. The sub-Hopf algebra of such functions is isomorphic to the Hopf algebra of functions on the fundamental group Λ/Λ_r. Such an f acts on λ by multiplication by $f(\lambda)$, so every such f will agree on λ and γ if and only if $\lambda - \gamma \in \Lambda_r$.

$\square$

Theorem 1. *$\lambda \sim \gamma$ if and only if $\lambda = \sigma \cdot \gamma$ for some $\sigma \in \mathcal{W}_l$.*

Proof: That the latter implies the former is exactly Proposition 1.

If $\lambda \sim \gamma$, then by Proposition 2 they are connected by an element of $\mathcal{W}^{\dagger}$. On the other hand by Lemma 4 they differ by an element of the root lattice, and thus the element of $\mathcal{W}^{\dagger}$ must preserve the root lattice (it is easy to see that if an element of $\mathcal{W}^{\dagger}$ takes one vector to another vector that differs from it by a root vector, the difference of any vector and its image is a root vector). Thus by Lemma 2 they are connected by an element of the affine Weyl group $\mathcal{W}_l$.

$\square$

Corollary 2. *In fact, $\{\chi_{\lambda}\}$, choosing one λ from each translated $\mathcal{W}_l$ equivalence class, is linearly independent as a set of functionals on the center.*

Proof: By Proposition 2 a linear relation between these would reduce to a linear relation between those in the translated $\mathcal{W}^{\dagger}$ orbit of some λ. Since elements of this orbit which are not $\mathcal{W}_l$ equivalent must be in distinct classes of Λ/Λ_r, computing them on the center intersected with $U_{\mathbf{s}}^{\dagger}(\mathfrak{h})$ shows that no such nontrivial relation exists. $\square$

3. Weyl Filtrations and Tilting Modules

In this section use U to refer to any of the forms of the quantum group defined in the last two sections: $U_q(\mathfrak{g})$, $U_{\mathcal{A}}^{\mathrm{res}}(\mathfrak{g})$, $U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$, $U_{\mathbf{q}}^{\mathrm{res}}(\mathfrak{g})$ or $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$, and use "the ground ring" to refer to $\mathbb{Q}(q)$, $\mathcal{A}$, $\mathcal{A}'$, $\mathbb{Q}[\mathbf{q}]$, or $\mathbb{Q}[\mathbf{s}]$ as appropriate. Also, drop the subscripts from such notation as $W_{\mathcal{A}'}^{\lambda}$ when no confusion would ensue. While many of the results of this section will apply to all forms of the quantum group, we will be interested in their application only to $U_{\mathbf{q}}^{\mathrm{res}}$ and $U_{\mathbf{s}}^{\dagger}$.

Definition 2. *a U-module V is said to have a Weyl filtration if there exists a sequence of submodules*
$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$
such that for each $1 \leq i \leq n$, V_i/V_{i-1} is isomorphic to the Weyl module W^{λ} for some $\lambda \in \Lambda^+$.

Proposition 3. *Suppose W is a $U_{\mathcal{A}}^{res}(\mathfrak{g})$-module such that $W \otimes_{\mathcal{A}} \mathbb{Q}(q) = \bigoplus_i W_q^{\lambda_i}$. Then $W \otimes_{\mathcal{A}} \mathbb{Q}[\mathbf{q}]$ and $(W \otimes_{\mathcal{A}} \mathcal{A}') \otimes_{\mathcal{A}'} \mathbb{Q}[\mathbf{s}]$ admit Weyl filtrations with the ith factor of highest weight λ_i, where the λ_i are assumed to be ordered so that λ_i is never greater than λ_j for $j < i$.*

Proof: We shall prove the proposition over $\mathbb{Q}[\mathbf{q}]$, the argument is exactly the same for $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$.

Decomposing $W = W_{\text{tor}} \oplus W_{\text{free}}$ into its torsion and free parts over $\mathcal{A}$, notice that $W \otimes_{\mathcal{A}} \mathbb{Q}(q) = W_{\text{free}} \otimes_{\mathcal{A}} \mathbb{Q}(q)$ and likewise for $\mathbb{Q}[\mathbf{q}]$. Since W_{tor} is a $U_{\mathcal{A}}^{\text{res}}$-module, the quotient by it is a free $\mathcal{A}$-module and a $U_{\mathcal{A}}^{\text{res}}$ module whose tensor with $\mathbb{Q}(q)$ and $\mathbb{Q}[\mathbf{q}]$ are isomorphic to that of W. Thus we can assume W is a free $\mathcal{A}$-module. Notice in this case the maps $v \mapsto v \otimes 1$ are injective maps from W to $W \otimes_{\mathcal{A}} \mathbb{Q}(q)$ and $W \otimes_{\mathcal{A}} \mathbb{Q}[\mathbf{q}]$ whose ranges span.

Let $w \in W$ be such that $w \otimes 1 \in W \otimes_{\mathcal{A}} \mathbb{Q}(q)$ is a vector of weight λ_1. By the maximality of λ_1 w must be a highest weight vector. Then $U_{\mathcal{A}}^{\text{res}} w$ is a $U_{\mathcal{A}}^{\text{res}}$-module, free over $\mathcal{A}$, whose tensor product with $\mathbb{Q}(q)$ yields a U_q-module isomorphic to $W_q^{\lambda_1}$. Thus $U_{\mathcal{A}}^{\text{res}} w$ must be isomorphic to $W_{\mathcal{A}}^{\lambda_1}$. Its tensor product with $\mathbb{Q}[\mathbf{q}]$ gives a submodule isomorphic to $W_{\mathbf{q}}^{\lambda_1}$. Therefore the quotient $W/W_{\mathcal{A}}^{\lambda_1}$ is a module whose tensor product with $\mathbb{Q}(q)$ is isomorphic to $\bigoplus_{i>1} W_q^{\lambda_i}$ and whose tensor product with $\mathbb{Q}[\mathbf{q}]$ is $(W \otimes_{\mathcal{A}} \mathbb{Q}[\mathbf{q}])/W_{\mathbf{q}}^{\lambda_1}$. By induction the proposition follows. $\qquad\square$

Corollary 3. *The tensor product of two $U_{\mathbf{q}}^{res}$ or $U_{\mathbf{s}}^{\dagger}$ modules with a Weyl filtration admits a Weyl filtration.*

Proof: By induction it suffices to prove that $W^{\lambda} \otimes W^{\gamma}$ admits a Weyl filtration. This follows from the previous proposition. $\qquad\square$

Remark 2. *Notice the entries in that Weyl filtration are the same as the entries in the classical decomposition of the tensor product of classical modules which were direct sums with the same entries as the original Weyl filtrations. Thus if we restrict attention to modules with a Weyl filtration the category of such modules forms a monoidal category which is "the same" as the tensor category of classical finite-dimensional modules if we replace the notion of direct sum decomposition of modules with that of Weyl filtration.*

Recall if V is a module over U the space of linear functionals from V to the ground ring is naturally a right U-module V^*. We can compose the induced representation with either S or S^{-1} to make it a left module which we also call V^* (the two ways of doing this give distinct but isomorphic module structures, and the distinction will not be relevant to us). If V is a finite direct sum of free finite-rank weight spaces, so is V^*. In this case, V is isomorphic as a U-module to V^{**} (though not by the canonical identification between these two as modules over the ground ring, instead this identification must be composed with conjugation by $q^{2\rho}$). The dual of the tensor product of two modules is isomorphic to the tensor product of the duals.

Definition 3. *a U-module V is said to have a* dual Weyl filtration *if there exists a sequence of submodules*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

such that for each $1 \le i \le n$ V_i/V_{i-1} is isomorphic to the dual of a Weyl module $(W^{\lambda})^*$ for some $\lambda \in \Lambda^+$.

Note that V admits a dual Weyl filtration if and only if V^* admits a Weyl filtration.

Definition 4. *A U-module V is a* tilting module *if it admits both a Weyl filtration and a dual Weyl filtration.*

Corollary 4. *The properties of admitting a Weyl filtration, admitting a good filtration or being a tilting module are preserved by tensor product.*

Remark 3. *The category of tilting modules forms a ribbon category (i.e. with tensor products and duals) which is not semisimple, but because of the existence of Weyl and dual Weyl filtrations, behaves in many respects like the semisimple tensor category of classical $\mathfrak{g}$ modules, in particular as far as the link invariant is concerned.*

If $V \cong W \bigoplus W'$, then V is tilting if and only if W and W' are tilting, so to understand tilting modules it suffices to understand indecomposable tilting modules. To do this requires a short detour into elementary homological algebra. For more detail on the subject, see Mac Lane [Mac63].

Recall that for two modules A and C over a ring, the set of exact sequences

$$0 \to C \to ? \to \cdots \to ? \to A \to 0$$

forms a chain complex indexed by the number of intervening modules, with an appropriate boundary operator. The associated homology $\text{Ext}^n(A, C)$ is functorial in each variable, and if

$$0 \to X \to Y \to Z \to 0$$

is a short exact sequence then we get the long exact sequences of homology

$$
(4) \quad
\begin{array}{l}
0 \longrightarrow \text{Ext}^0(Z, C) \longrightarrow \text{Ext}^0(Y, C) \longrightarrow \text{Ext}^0(X, C) \longrightarrow \\[4pt]
\quad \text{Ext}^1(Z, C) \longrightarrow \text{Ext}^1(Y, C) \longrightarrow \quad \cdots
\end{array}
$$

and

$$
(5) \quad
\begin{array}{l}
0 \longrightarrow \text{Ext}^0(A, X) \longrightarrow \text{Ext}^0(A, Y) \longrightarrow \text{Ext}^0(A, Z) \longrightarrow \\[4pt]
\quad \text{Ext}^1(A, X) \longrightarrow \text{Ext}^1(A, Y) \longrightarrow \quad \cdots.
\end{array}
$$

Finally, $\text{Ext}^0(A, C) \cong \text{Hom}(A, C)$ and if A and C are such that every short exact sequence $0 \to C \to B \to A \to 0$ splits, then $\text{Ext}^1(A, C) = 0$.

Lemma 5. *If V is a U-module and $\lambda \in \Lambda^+$ is such that no weight appearing in V is greater than λ then any quotient*

$$V \xrightarrow{\;g\;} W^\lambda \longrightarrow 0$$

splits.

Proof: There must be a vector $v \in V$ of weight λ which is in the preimage of a highest weight vector in W^λ under g. By the condition on λ, v is of maximal weight in V and hence is a highest weight vector, so there is a map $g' : W^\lambda \to V$ sending a highest weight vector to v. Clearly gg' is nonzero on the highest weight vector and hence is a multiple of 1, so the sequence splits. $\qquad\square$

Lemma 6. *If A admits a Weyl filtration, and C admits a dual Weyl filtration then* $\mathrm{Ext}^1(A, C) = 0$.

Proof: We will prove first the base case, then do induction on the filtration of A, then induction on the filtration of C.

- $\mathrm{Ext}^1(W^\lambda, (W^\gamma)^*) = 0$ *for all* $\lambda, \gamma \in \Lambda^+$. Suppose first that $\lambda \not< \gamma^*$, where γ^*, which is minus the image of γ under the action of the longest element of the Weyl group, is the maximal highest weight of $(W^\gamma)^*$. Then if
$$0 \to (W^\gamma)^* \to B \to W^\lambda \to 0$$
  the sequence splits by Lemma 5 and the result follows. On the other hand if $\lambda < \gamma^*$ and
$$0 \to (W^\gamma)^* \to B \to W^\lambda \to 0$$
  then dualizing
$$0 \to (W^\lambda)^* \to B^* \to W^\gamma \to 0$$
  and again by Lemma 5 the result follows.
- $\mathrm{Ext}(A, (W^\gamma)^*) = 0$ *if A admits a Weyl filtration.* By induction there is a short exact sequence
$$0 \to W^\lambda \to A \to A' \to 0$$
  with A' admitting a Weyl filtration and hence $\mathrm{Ext}^1(A', (W^\gamma)^*) = 0$. From the long exact sequence (4)
$$\to [\mathrm{Ext}^1(W^\lambda, (W^\gamma)^*) = 0] \to \mathrm{Ext}^1(A, (W^\gamma)^*) \to [0 = \mathrm{Ext}^1(A', (W^\gamma)^*)] \to$$
  from which it follows $\mathrm{Ext}(A, (W^\gamma)^*) = 0$.
- $\mathrm{Ext}(A, C) = 0$ *if A admits a Weyl filtration and C admits a dual Weyl filtration.* Again inductively we have a sequence
$$0 \to (W^\gamma)^* \to C \to C' \to 0$$
  with C' admitting a dual Weyl filtration and hence $\mathrm{Ext}^1(A, C) = 0$. Again the long exact sequence 5 and the previous two items give $\mathrm{Ext}(A, C) = 0$.

$\square$

Proposition 4. *If Q, Q' are indecomposable tilting modules over U each with a maximal vector of weight λ, then $Q \cong Q'$.*

Proof: Suppose v is a weight λ vector in Q and v' is a weight λ vector in Q'. Let $f : W^\lambda \to Q$ and $f' : W^\lambda \to Q'$ send a particular highest weight vector to v and v' respectively. Let j be the smallest integer such that V_j contains v in a Weyl filtration of Q. Then $V_j/V_{j-1} \cong W^\lambda$. By the maximality of λ, $V_{j-1}/V_{j-2} \cong W^\gamma$ with $\lambda \not< \gamma$. By Lemma 5 we can find a new V'_{j-1} (without changing V_{j-2}) such that the filtration is still Weyl but v is now an element of V_{j-1}. Inductively, there exists a Weyl filtration with the image of f being V_1, which is to say there is a short exact sequence
$$0 \longrightarrow W^\lambda \xrightarrow{\ f\ } Q \longrightarrow N \longrightarrow 0$$

with N admitting a Weyl filtration. The long exact sequence (4) gives

$$0 \to \mathrm{Hom}(N, Q') \to \mathrm{Hom}(Q, Q') \to \mathrm{Hom}(W^\lambda, Q') \to [0 = \mathrm{Ext}^1(N, Q')] \to$$

by Lemma 6, so that the sequence is in fact short exact, and $f' \in \mathrm{Hom}(W^\lambda, Q')$ must factor through a map $g' : Q \to Q'$ which takes v to a nonzero multiple of v'. By the same argument with Q and Q' reversed there is a map $g : Q' \to Q$ taking v' to a nonzero multiple of v. Thus gg' is a map from Q to itself taking v to a nonzero multiple of itself.

$\{(gg')^n[Q]\}_{n \in \mathbb{N}}$ is a nested sequence of submodules and thus by finite-dimensionality must stabilize on some submodule $(gg')^M[Q]$ such that gg' is onto when restricted to this submodule. So $Q = (gg')^M[Q] \bigoplus \mathrm{Ker}((gg')^M)$. Since Q is indecomposable one of these summands must be zero, and since $v \in (gg')^M Q$ one has $Q = (gg')^M[Q]$ so gg' is invertible. $\square$

Corollary 5. *Eery tilting module is a direct sum of indecomposable tilting modules. Every indecomposable tilting module is isomorphic to some T_λ, the unique indecomposable indecomposable tilting module with a maximal vector of weight λ.*

Corollary 6 (Linkage Principle). *A simple module with highest weight λ can occur as a composition factor in the Weyl or indecomposable tilting module of highest weight γ only if $\lambda \leq \gamma$ and $\lambda = \sigma \cdot \gamma$ for some $\sigma \in \mathcal{W}_l$.*

4. Negligible Modules and the Weyl Alcove

As in Section 2 we work with $\mathbf{s}$ a primitive lLth root of unity, and consider the quantum group $U_\mathbf{s}^\dagger(\mathfrak{g})$. Let M be the lattice of translations $l'\check{\Lambda}_r$ or $l'\Lambda_r$ according to whether D divides l' or not, so that $\mathcal{W}_l = \mathcal{W} \ltimes M$. Define hyperplanes

$$w_{k,\alpha} = \begin{cases} \{x \in \mathfrak{h}^*, \, \langle x + \rho, \alpha \rangle = kl'\} & \text{if } D|l \\ \{x \in \mathfrak{h}^*, \, \langle x + \rho, \check{\alpha} \rangle = kl'\} & \text{else,} \end{cases}$$

for $\alpha \in \Phi^+$, called the *walls* of the (translated) affine Weyl group, so that the translated action of $\mathcal{W}_l$ is generated by reflections $\sigma_{k,\alpha}$ about $w_{k,\alpha}$. These hyperplanes divide $\mathfrak{h}^*$ into compact regions, called *alcoves,* including the principal alcove C_l, such that $\sigma \mapsto \sigma \cdot C_l$ is a bijection between elements of $\mathcal{W}_l$ and alcoves. The hyperplanes are called the *walls* of the alcoves. Likewise the *walls* of $\mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$ are the hyperplanes $\langle x + \rho, \alpha \rangle = kl/2$. If x is on the wall $w_{k,\alpha}$ and no other wall then the stabilizer of x in $\mathcal{W}_l$ is $\{1, \sigma_{k,\alpha}\}$. A wall $w_{k,\alpha}$ of an alcove $\sigma \cdot C_l$ is called a *lower wall* if every point y in the interior satisfies $\langle y + \rho, \alpha \rangle$ is greater than the corresponding quantity for the points on the wall, and an *upper wall* otherwise. Finally, let θ_0 be θ if D divides l' and ϕ otherwise, so that w_{1,θ_0} is the unique upper wall of C_l, and the intersection of the interior of C_l with Λ is

$$\Lambda^l \stackrel{\text{def}}{=} \{\lambda \in \Lambda^+ \, | \, \langle \lambda + \rho, \theta_0 \rangle < l'\}.$$

A module V is called *negligible* if every intertwiner $\phi \colon \to V$ has quantumm trace 0.

Theorem 2. *In $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$, every T_λ with λ not in Λ^l is negligible, provided $l' \geq D\check{h}$ if $D|l'$ or $l' > h$ otherwise.*

Lemma 7. *an indecomposible titling module is negligible if and only if its quantum dimension is zero.*

Proof: The algebra of intertwiners from an indecomposable module to itself consists of multiples of the identity and nipotent intertwiners. Since intertwiners commute with $q^{2\rho}$, they hae quantum trace zero. If the module has quantum dimension zero, then multiples of the identity hae quantum trace zero. $\qquad\square$

Lemma 8. *The tensor product of a neglible module and another module is a direct sum of negligible indecomposable modules.*

Proof: If V is a negligible module and W is not, then the intertwiners on $V \otimes W$ are just the tensor product of the set of intertiners on V tensored with those on W. Thus the quantum trace of any such intertwiner is the sum of a product of quantum traces of intertwiners on V and W. In each term the first entry of the product is zero.

If a module is negligible, so are all its direct summands. $\qquad\square$

Lemma 9.

(a) *If $\sigma \in \mathcal{W}^{\dagger}$, then $\mathrm{qdim}(\lambda) = (-1)^{\sigma}\,\mathrm{qdim}(\sigma \cdot \lambda)$ whenever $\lambda, \sigma \cdot \lambda \in \Lambda^+$, where $(-1)^{\sigma}$ represents the orientation of σ. In particular this is true of $\sigma \in \mathcal{W}_l$. Further, $\mathrm{qdim}(\lambda) = 0$ if and only if λ has nontrivial stabilizer in $\mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$.*
(b) *Every T_λ where λ is on a wall of $\mathcal{W}_l$ is negligible.*

Proof:

(a) By the Weyl formula (2), $\mathrm{qdim}(\lambda)$ is
$$\mathrm{qdim}(\lambda) = \prod_{\beta > 0} \left(\mathbf{q}^{\langle\lambda+\rho,\beta\rangle} - \mathbf{q}^{-\langle\lambda+\rho,\beta\rangle} \right) / \left(\mathbf{q}^{\langle\rho,\beta\rangle} - \mathbf{q}^{-\langle\rho,\beta\rangle} \right).$$

In fact we can interpret $\mathrm{qdim}(\lambda)$ by this formula even when λ is not in Λ^+. It suffices to prove the first sentence when σ is a generator of the classical Weyl group $\sigma_{0,\alpha_i} = \sigma_{\alpha_i}$ and when σ is translation by $l\check{\theta}/2$.

Suppose first that σ is σ_{α_i}, then

$$\mathrm{qdim}(\sigma \cdot \lambda) = \prod_{\beta > 0} \left(\mathbf{q}^{(\sigma \cdot \lambda + \rho, \beta)} - \mathbf{q}^{-(\sigma \cdot \lambda + \rho, \beta)} \right) / \left(\mathbf{q}^{(\rho,\beta)} - \mathbf{q}^{-(\rho,\beta)} \right)$$

$$= \prod_{\beta > 0} \left(\mathbf{q}^{(\sigma_{\alpha_i}(\lambda+\rho),\beta)} - \mathbf{q}^{(-\sigma_{\alpha_i}(\lambda+\rho),\beta)} \right) / \left(\mathbf{q}^{(\rho,\beta)} - \mathbf{q}^{-(\rho,\beta)} \right)$$

$$= \prod_{\beta > 0} \left(\mathbf{q}^{(\lambda+\rho,\sigma_{\alpha_i}(\beta))} - \mathbf{q}^{-(\lambda+\rho,\sigma_{\alpha_i}(\beta))} \right) / \left(\mathbf{q}^{(\rho,\beta)} - \mathbf{q}^{-(\rho,\beta)} \right)$$

since σ_{α_i} is a unipotent isometry. Notice that σ_{α_i} permutes the positive roots of Φ except for α_i which it reverses ([Hum72][10.2]) so all factors above stay the same except for one which changes sign. Thus the formula above gives $-\mathrm{qdim}(\sigma \cdot \lambda) = (-1)^{\sigma}\,\mathrm{qdim}(\lambda)$.

Now suppose σ is translation by $l\check{\theta}/2$. Then

$$\mathrm{qdim}(\sigma \cdot \lambda) = \prod_{\beta>0} \left(\mathbf{q}^{(\sigma \cdot \lambda + \rho, \beta)} - \mathbf{q}^{-(\sigma \cdot \lambda + \rho, \beta)} \right) / \left(\mathbf{q}^{(\rho,\beta)} - \mathbf{q}^{-(\rho,\beta)} \right)$$

$$= \prod_{\beta>0} \left(\mathbf{q}^{(l\check{\theta}/2 + \lambda + \rho, \beta)} - \mathbf{q}^{-(l\check{\theta}/2 + \lambda + \rho, \beta)} \right) / \left(\mathbf{q}^{(\rho,\beta)} - \mathbf{q}^{-(\rho,\beta)} \right)$$

$$= \prod_{\beta>0} \left(\mathbf{q}^{l\langle \beta, \check{\theta}\rangle/2} \mathbf{q}^{(\lambda+\rho,\beta)} - \mathbf{q}^{-l\langle \beta, \check{\theta}\rangle/2} \mathbf{q}^{-(\lambda+\rho,\beta)} \right) / \left(\mathbf{q}^{(\rho,\beta)} - \mathbf{q}^{-(\rho,\beta)} \right)$$

$$= \left(\prod_{\beta>0} \mathbf{q}^{l\langle \beta, \check{\theta}\rangle/2} \right) \prod_{\beta>0} \left(\mathbf{q}^{(\lambda+\rho,\beta)} - \mathbf{q}^{-(\lambda+\rho,\beta)} \right) / \left(\mathbf{q}^{(\rho,\beta)} - \mathbf{q}^{-(\rho,\beta)} \right)$$

$$= \mathbf{q}^{l\langle \rho, \check{\theta}\rangle} \mathrm{qdim}(\lambda)$$

$$= \mathrm{qdim}(\lambda).$$

Since the affine Weyl group is a subgroup of $\mathcal{W}^\dagger$, the same result applies to the smaller group.

Of course if λ has nontrivial stabilizer than it lies on some wall so there is a reflection σ which fixes λ and $\mathrm{qdim}(\lambda) = \mathrm{qdim}(\sigma \cdot \lambda) = -\mathrm{qdim}(\lambda)$, so $\mathrm{qdim}(\lambda) = 0$. λ has no stabilizer $\mathrm{qdim}(\lambda)$ is a product of nonzero quantities, and thus nonzero.

(b) By the Linkage Principle, Corollary 6, T_λ has a Weyl filtration all of whose entries are affine Weyl equivalent to λ. If λ is on a wall, so are all weights in its affine orbit, and hence the quantum dimension of T_λ, which is a sum of the quantum dimensions of the entries of the Weyl filtration, is zero. Therefore T_λ is negligible.

$\square$

Corollary 7. *As functionals on the center*

$$\mathrm{qtr}_{\sigma \cdot \lambda} = (-1)^\sigma \mathrm{qtr}_\lambda$$

when $\sigma \in \mathcal{W}_l$. As functionals on the image of quantum traces under Ψ the same is true when $\sigma \in \mathcal{W} \ltimes \frac{l}{2}\check{\Lambda}_r$ and $\sigma \cdot \lambda \in \Lambda$.

Lemma 10.

(a) *If W_μ appears in a Weyl filtration of a tilting module T, and no μ' in the translated $\mathcal{W}_l$ orbit of μ with $\mu' > \mu$ appears in that filtration, then T_μ is a direct summand of T.*

(b) *If μ appears as a highest weight in the* classical *module $W^\lambda \otimes W^\gamma$ and no $\mu' > \mu$ in the translated $\mathcal{W}_l$ orbit of μ appears in the classical module $W^{\lambda'} \otimes W^{\gamma'}$ for $\lambda' \leq \lambda$ and $\gamma' \leq \gamma$ in the translated $\mathcal{W}_l$ orbits of λ and γ respectively, then T_μ is a direct summand of $T_\lambda \otimes T_\gamma$.*

Proof:

(a) By the Linkage Principle, Corollary 6, W_μ appears in the filtration of an indecomposable direct summand whose Weyl decomposition contains only modules with highest weights in the translated $\mathcal{W}_l$ orbit of μ. By the assumption on μ the weight μ is maximal in this summand, which must thus be isomorphic to T_μ by Proposition 4.

(b) Of course a factor of W_μ must appear in a filtration of $T_\lambda \otimes T_\gamma$ by Proposition 3 and Corollary 3. If a larger μ' in the orbit of μ also appeared in the filtration, it would appear in the classical decomposition of some $W^{\lambda'} \otimes W^{\gamma'}$ with λ' and γ' in a Weyl filtration of T_λ and T_γ respectively. This is ruled out by the assumption, so by part (a) we are done.

$\square$

Lemma 11. *Suppose $\lambda, \gamma, \lambda + \sigma(\gamma) \in \Lambda^+$ for some σ in the classical Weyl group $\mathcal{W}$, $\gamma \in \Lambda^l$, suppose λ is on exactly one wall $w_{k,\alpha}$ and $\lambda + \sigma(\gamma)$ is in the interior of an alcove for which $w_{k,\alpha}$ is a lower wall. Then $T_{\lambda+\sigma(\gamma)}$ is a direct summand of $T_\lambda \otimes T_\gamma$.*

Proof: By Lemma 10(b), we must check that $\lambda + \sigma(\gamma)$ occurs as a highest weight in the classical decomposition of $W^\lambda \otimes W^\gamma$, and that nothing greater in its $\mathcal{W}_l$ orbit occurs as a highest weight in classical $W^{\lambda'} \otimes W^\gamma$ with λ' $\mathcal{W}_l$-equivalent to and less than λ (nothing is $\mathcal{W}_l$-equivalent to and less than γ because it is in the Weyl alcove).

For the first point, consider the classical Racah formula, Equation (10). Note that the result is true unless $\lambda + \sigma(\gamma)$ is the result of the translated action of a nontrivial element τ of the classical Weyl group on $\lambda + \mu$ for some μ that occurs as a weight of W^γ. It is easy to see that if λ, λ' are in the Weyl chamber then $\tau \cdot \lambda'$ is strictly further from λ than λ' for any $\tau \in \mathcal{W}$, so the length of μ must be strictly greater than the length of $\sigma(\gamma)$, which is not possible if μ is a weight of W^γ.

Essentially the same argument applies for the second point. Since $\lambda, \lambda + \sigma(\gamma)$ are in the same alcove, any λ', μ' in the translated $\mathcal{W}_l$ orbits respectively of λ and $\lambda + \sigma(\gamma)$ must be at least as far away from each other as $\lambda + \sigma(\gamma)$ and λ are, with equality only when μ', λ' are in the same alcove. But if μ' is a weight in classical $W^{\lambda'} \otimes W^\gamma$, it must be at most $||\gamma||$ away from λ', with that distance only achieved if $\mu' - \lambda' = \sigma'(\gamma)$ for some $\sigma' \in \mathcal{W}$. Thus if $\lambda' < \lambda$, λ', μ' are in the orbits of λ and $\lambda + \sigma(\lambda)$, and μ' is in $W^\lambda \otimes W^\gamma$, then μ' is in the same alcove as λ', so there is a single $\tau \in \mathcal{W}_l$ such that $\tau \cdot \lambda = \lambda'$ and $\tau \cdot (\lambda + \sigma(\lambda)) = \mu'$. If $\mu' > \lambda + \sigma(\gamma)$, then $\lambda' \geq \lambda$, so we must have $\lambda = \lambda'$. In this case $\tau = \sigma_{n,\alpha}$, the reflection about the wall on which λ lies. since λ is on a lower wall this would make $\mu' \leq \lambda + \sigma(\gamma)$. Thus by contradiction the result is proven. $\square$

Proof: [Of Theorem 2] In light of Lemma 11 and Lemma 9, it suffices to find for each alcove other than C_l with nonempty intersection with Λ^+ a dominant weight λ on the interior of a lower wall of that alcove. Then each μ in the interior of this alcove, since $\mu - \lambda$ is Weyl conjugate to something in C_l, would have a T_μ as a summand in some $T_\lambda \otimes T_\gamma$, and thus would be negligible. This requires that every such alcove have a lower wall whose intersection with the weight lattice consists of dominant weights, and that on the interior of each wall of the fundamental alcove there is a weight.

For the first, notice that every wall of an alcove is either a wall of the principal chamber for the translated action of the classical Weyl group or is transverse to it, so every wall of every alcove either contains no dominant weights or all the weights in its interior are dominant. If the alcove intersects Λ^+ all of its walls that do not intersect Λ^+ must be part of the walls of the chamber. If all the lower walls of an alcove are walls of the chamber, the alcove clearly must be C_l.

For the second, if the the wall is w_{0,α_i}, one can readily check that $-\lambda_i$ lies in the interior of the wall (under the restriction on l). If the wall is $w_{1,\theta}$ ($D|l'$ case), notice there is always a fundamental weight λ_i such that $\langle \lambda_i, \check{\theta} \rangle = 1$ (Check [Hum72][p. 66]), so $(l' - \check{h}/D)\lambda_i$ lies on $w_{1,\theta}$. Since it is a dominant weight it lies on no other walls. If the wall is $w_{1,\phi}$ ($D \nmid l'$ case), we can find λ_i such that $\langle \lambda_i, \phi \rangle = 1$ for B_n and C_n, and therefore $(l' - h)\lambda_i$ will do the trick. There remains only G_2 and F_4 to consider.

For G_2, we check that $\langle \lambda_1, \phi \rangle = 2$, $\langle \lambda_2, \phi \rangle = 3$. Now every integer greater than 1 can be written as a nonnegative integer combination of 2 and 3, and every number greater than 6 can be written so with neither coefficient equal to zero. Thus if $l > 6 = h$ there exists a positive integer combination of λ_1 and λ_2 whose inner product with ϕ is l. Thus this integer combination minus ρ lies on $w_{l,\phi}$ and no other wall.

For F_4, we check that $\langle \lambda_i, \phi \rangle$ gives $2, 4, 3, 2$ for $i = 1 \ldots 4$. Again if $l > 12 - h$, then l can be written as a positive linear combination of these four numbers, and thus the same combination of λ_1 through λ_4 gives a weight on the interior of $w_{l,\phi}$. $\qquad \square$

5. The Quantum Racah Formula

For classical Lie algebras or generic q write

$$(6) \qquad W^\lambda \otimes W^\gamma \cong \bigoplus_{\mu \in \Lambda^+} N^\mu_{\lambda,\gamma} W^\mu$$

where $N^\mu_{\lambda,\gamma}$ are nonnegative integers representing multiplicities.

For $\mathbf{q}$ an lth root of unity, if $\lambda, \gamma \in \Lambda^l$, then

$$W^\lambda \otimes W^\gamma \cong \bigoplus_{\mu \in \Lambda^l} M^\mu_{\lambda,\gamma} W^\mu \oplus N$$

where N is a negligible tilting module and each $M^\mu_{\lambda,\gamma}$ is a nonnegative integer representing multiplicity (because W^λ, W^γ, and W^μ are all titling modules). We define the *truncated tensor product* $\hat{\otimes}$ on direct sums of Weyl modules in Λ^l by extending the following to direct sums:

$$(7) \qquad W^\lambda \hat{\otimes} W^\gamma = W^\lambda \otimes W^\gamma / N \cong \bigoplus_{\mu \in \Lambda^l} M^\mu_{\lambda,\gamma} W^\mu.$$

We will see in the next section that this gives a monoidal structure on the category of such modules.

Proposition 5.

$$(8) \qquad M^{\mu}_{\lambda,\gamma} = \sum_{\mu \in \Lambda^l} \sum_{\substack{\sigma \in \mathcal{W}_l \\ \sigma \cdot \mu \in \Lambda^+}} (-1)^{\sigma} N^{\sigma \cdot \mu}_{\lambda,\gamma}.$$

Proof: Over $\mathcal{A}$,

$$\mathrm{qtr}_{W^{\lambda} \otimes W^{\gamma}} = \sum_{\mu} N^{\mu}_{\lambda,\gamma} \, \mathrm{qtr}_{W^{\mu}}$$

so in particular the same holds over $\mathbb{Q}[\mathbf{s}]$. As a functional on the center this is equal to

$$\sum_{\mu \in \Lambda^l} \sum_{\substack{\sigma \in \mathcal{W}_l \\ \sigma \cdot \mu \in \Lambda^+}} (-1)^{\sigma} N^{\sigma \cdot \mu}_{\lambda,\gamma}.$$

On the other hand as a functional on the center

$$\mathrm{qtr}_{W^{\lambda} \otimes W^{\gamma}} = \mathrm{qtr}_{W^{\lambda} \hat{\otimes} W^{\gamma}} = \sum_{\mu \in \Lambda^l} M^{\mu}_{\lambda,\gamma} \, \mathrm{qtr}_{\mu}.$$

Since $\{\mathrm{qtr}_{\mu}\}_{\mu \in \Lambda^l}$ are linearly independent as functionals on the center (Corollary 2), the result follows. $\qquad\square$

Corollary 8 (Quantum Racah Formula)**.**

$$(9) \qquad M^{\mu}_{\lambda,\gamma} = \sum_{\sigma \in \mathcal{W}_l} (-1)^{\sigma} \dim(W^{\lambda}(\sigma \cdot \mu - \gamma))$$

where $W^{\lambda}(\gamma)$ is the subspace of W^{λ} of weight γ.

Proof: This result relies on the *classical Racah formula,* which says that

$$(10) \qquad N^{\mu}_{\lambda,\gamma} = \sum_{\tau \in \mathcal{W}} (-1)^{\tau} \dim(W^{\lambda}(\tau \cdot \mu - \gamma)).$$

Recall that Λ^+ is a fundamental domain for the action of $\mathcal{W}$ and that only the identity fixes it. Suppose $\sigma \in \mathcal{W}_l$ takes the principal Weyl alcove C_l to some domain C. There is a unique element $\tau \in \mathcal{W}$ such that τ^{-1} of C intersects Λ^+. Thus $\tau^{-1}\sigma$ takes Λ^l to some fundamental domain in Λ^+. We conclude that every element of the affine Weyl group can be written uniquely as $\tau\eta$, where $\tau \in \mathcal{W}$ and $\eta[\Lambda^l] \subset \Lambda^+$. Thus

$$M^{\mu}_{\lambda,\gamma} = \sum_{\eta[\Lambda^l] \subset \Lambda^+} (-1)^{\eta} N^{\eta \cdot \mu}_{\lambda,\gamma}$$

$$= \sum_{\eta[\Lambda^l] \subset \Lambda^+} (-1)^{\eta} \sum_{\tau \in \mathcal{W}} (-1)^{\tau} \dim(W^{\lambda}(\tau\eta \cdot \mu - \gamma)$$

$$= \sum_{\sigma \in \mathcal{W}_l} (-1)^{\sigma} \dim(W^{\lambda}(\sigma \cdot \mu - \gamma).$$

$$\square$$

Remark 4. *The quantum Racah formula (9), just like the classical version, admits a beautiful concrete algorithm for the computation of $M_{\lambda,\gamma}^{\mu}$ in rank 2 which illustrates its geometric flavor. Draw the weight lattice. Cover this with a piece of tracing paper and mark off next to each weight the dimension of the corresponding weight space of V^{γ}. Now slide the tracing paper so that what initially covered the the 0 weight space now lies over the weight space λ. Fold the tracing paper along the walls of C_l, and continue folding until the paper fits within it. For each weight μ add up all the numbers that now lie over the point μ, subtracting those numbers that appear in reverse (write with a sufficiently seriphed font that you can distinguish them!). This sum is $M_{\lambda,\gamma}^{\mu}$.*

Corollary 9. *Let ι be an isometry of C_l which preserves weights (hence also an isometry of Λ^l) given by a translation composed with the translated action of a Weyl group element. Then for all $\lambda, \gamma, \mu \in \Lambda^l$,*

$$M_{\lambda,\gamma}^{\mu} = M_{\lambda,\iota(\gamma)}^{\iota(\mu)}.$$

Proof: Write $\iota(\gamma) = \sigma \cdot \gamma + t$, where $\sigma \in \mathcal{W}$ and t is a weight. Notice $\sigma(\lambda - \gamma) = \iota(\lambda) - \iota(\gamma)$. Also, since $\iota(-\rho) = t + \rho$ is a vertex of C_l, $\langle t, \check{\alpha}_i \rangle = l_i'$, which is to say $\sigma_{0,\alpha_i}(t) - t = l_i'\alpha_i \in M$ for all simple $\alpha_i \in \Delta$. Likewise $\sigma_{1,\theta_0}(t) - t \in M$ (check on translation by $l'\theta_0$) so $\sigma(t) - t \in M$ for all $\sigma \in \mathcal{W}_l$. Therefore

$$\iota^{-1}\sigma'\iota \in \mathcal{W}_l \qquad \text{for all } \sigma' \in \mathcal{W}_l.$$

Then

$$M_{\lambda,\gamma}^{\mu} = \sum_{\sigma' \in \mathcal{W}_l} (-1)^{\sigma'} \dim(W^{\lambda}(\sigma' \cdot \mu - \gamma))$$

$$= \sum_{\sigma' \in \mathcal{W}_l} (-1)^{\sigma'} \dim(W^{\lambda}(\iota(\sigma' \cdot \mu) - \iota(\gamma)))$$

$$= \sum_{\sigma' \in \mathcal{W}_l} (-1)^{\sigma'} \dim(W^{\lambda}(\sigma'' \cdot \iota(\mu) - \iota(\gamma)))$$

$$= M_{\lambda,\iota(\gamma)}^{\iota(\mu)}$$

where σ'' is the conjugate of σ' by ι, which of course has the same orientation as σ'. $\qquad\square$

6. Ribbon Categories and Modularity

Theorem 3. *The category of modules of $U_{\mathcal{A}'}^{\dagger}(\mathfrak{g})$ which are free and finite-dimensional over $\mathcal{A}'$ is an Abelian ribbon category enriched over $\mathcal{A}'$. In particular the invariant of a link with components labeled by finite-rank modules is a polynomial in s with integer coefficients.*

Proof: Kassel in [Kas95][XI-XIV] defines ribbon categories and proves that the category of finite-dimensional modules of a ribbon Hopf algebra over a field forms a ribbon category. One can easily check that the proof goes through unchanged for topological Hopf algebras, and for Hopf algebras over a p.i.d., provided we restrict to

free modules (the freeness is required to define a map from the trivial module to the tensor product of a module with its dual). The last remark is simply the observation that the trivial object in the category is the module $\mathcal{A}'$ with the counit as action. $\qquad\square$

Remark 5. *Le in [Le00] has proven a stronger result than the last sentence. The link invariant (when all labels are Weyl modules) consists of a term depending only on the linking matrix and labels times an integer polynomial in q^2 and q^{-2}. In fact with care a similar result holds for the entire ribbon category.*

Theorem 4. *The category of all finite-dimensional $U_{\mathbf{s}}^{\dagger}(\mathfrak{g})$-modules is an Abelian ribbon category enriched over $\mathbb{Q}[\mathbf{s}]$, and the full subcategory of tilting modules is a ribbon subcategory. The invariant $I(L)$ of any labeled link L with a component labeled by W^{λ} is related to that of L', the same link with that component labeled by $W^{\sigma \cdot \lambda}$ for $\sigma \in \mathcal{W}_l$, by*

$$I(L') = (-1)^{\sigma} I(L).$$

Proof: Again the first sentence is a corollary of Kassel's proof. A subcategory of a ribbon category is ribbon so long as it is closed under tensor product and left duals, and this is the content of Corollary 4 (together with the obvious fact that the set of tilting modules is closed under taking duals).

It is shown for example by Kauffman and Radford [KR95] that to a one-tangle with components labeled by invariant functionals on a ribbon Hopf algebra there is associated an element of the center such that, if all the functionals are quantum traces of modules W_i and the quantum trace of some module V is applied to the action of this central element on V, the result is the ordinary link invariant of the link labeled by the given modules, with the open component closed and labeled by V. In other words the invariant is $\text{qdim}(V)\chi_V(z)$, where V is the label of the open component and z is the element of the center. The result follows from this fact together with Corollary 7. $\qquad\square$

Theorem 5. *The category of tilting modules has a full ribbon functor to a semisimple ribbon category $\mathcal{C}$ whose nonisomorphic simple objects are the image of the tilting modules with highest weight in Λ^l. The invariant of a ribbon spin network with labels in the original category is the same as the invariant of the same network labeled with the functorial image of those labels.*

Proof. We use the quotient construction of Mac Lane [Mac71][II.8]. Specifically, if $f, g \in \text{Hom}(V, W)$, we say that $f \sim g$ if, for all $h \in \text{Hom}(W, V)$, $\text{qtr}_V(hf) = \text{qtr}_V(hg)$. Such an equivalence relation defines a functor to a quotient category $\mathcal{C}$ such that $f \sim g$ implies f and g are sent by the functor to the same thing, and $\mathcal{C}$ is universal for this property.

It is clear that the image of any negligible tilting modules is null, i.e. isomorphic to the null module $\{0\}$. since $\text{qdim}(\lambda) \neq 0$ for all $\lambda \in \Lambda^l$, each such module is mapped to a non-null object and thus $\mathcal{C}$ is semisimple with these as simple objects.

Because $f \sim g$ implies $f \otimes h \sim g \otimes h$ and $h \otimes f \sim h \otimes g$ for all h, $\mathcal{C}$ inherits a tensor product structure making the functor a tensor functor. The image of the braiding

TABLE 1. Instances of extra symmetry in the Weyl alcove

	A_n	B_n	C_n	D_n	E_7
l odd	$\lambda_{\frac{n+1}{2}}$, if $2 \nmid n$	λ_n if $2\mid n$	λ_1	λ_1 if $2 \nmid n$ $\lambda_1, \lambda_{n-1}, \lambda_n$ if $2\mid n$	λ_7
l even		λ_n if $D \nmid l'$, $2 \nmid n$	λ_1 if $D \nmid l'$		

morphisms, duality morphisms and twist morphisms are braiding, duality and twist morphisms for $\mathcal{C}$.

$\square$

Remark 6. *Since each tilting module can be written uniquely as a direct sum $M \oplus N$, where M is isomorphic to a direct sum of modules in Λ^l and N is negligible, we can define an isomorphic functor from $\mathcal{C}$ to the full subcategory of the tilting module category consisting of modules isomorphic to a direct sum of modules in Λ^l. Unfortunately this functor does not preserve the tensor product. However this is a tensor isomorphism if the range category uses the truncated tensor product $\hat{\otimes}$ for its monoidal structure. This proves the truncated tensor product is a monoidal structure, and that in particular each $M^\mu_{\lambda,\gamma}$ is nonnegative.*

To address the modularity of $\mathcal{C}$ we must come to terms with the relationship between $\mathcal{W}_l$, the symmetries of the characters of Weyl modules, and $\mathcal{W}^\dagger$, the symmetries of the S-matrix. Actually, since we are only interested in the S-matrix applied to weights, the relevant group is the subgroup $\mathcal{W}_l^\Lambda \subset \mathcal{W}^\dagger$ which map weights to weights.

Proposition 6.
$\frac{l}{2}\check{\Lambda}_r \cap \Lambda$ *is the lattice generated by M, together with the vectors $l'\lambda_i$, where λ_i is as in Table 1, using the conventions in* [Hum72]*[Ch. 13] for the naming of fundamental weights (when no λ_i is given the lattice is exactly M). Notice each λ_i corresponds to an element of the fundamental group of order 2, though not every such occurs on the list.*

Proof: Suppose first that l is odd, and suppose $\check{\alpha} \in \check{\Lambda}_r$ such that $l\check{\alpha}/2 \in \Lambda$ but is not in M. Since $D\check{\alpha} \in \Lambda_r$ we can subtract any multiple of it from $l\check{\alpha}/2$ and it will still be in the weight lattice and still not be in M. As long as $D \neq 3$ this means $\check{\alpha}/2 \in \Lambda$, is half a coroot, and l times it is not in M. If $D|l$ this means it is a weight λ such that 2λ is a coroot but $l\lambda$ is not a coroot. Since l is odd this is equivalent to saying λ is not a coroot. Since the set of λ with these properties form a Λ_r equivalence class, it suffices to check which minimal λ_i are half a coroot but not a coroot themselves. If D does not divide l the same reasoning says we are looking for minimal λ_i such that $2\lambda_i$ is a coroot but $l\lambda_i$ is not a root. Again assuming D is not 3, we see that $l\lambda_i$ is a root if and only if λ_i is a root. The table lists those λ_i which are half a coroot but not a root.

the case $D = 3$, G_2, is handled similarly.

If l is even, the two lattices are equal unless D does not divide l'. If $l'\check{\alpha}$ is a weight, then by subtracting an appropriate multiple of $D\check{\alpha}$ we see that $\check{\alpha}$ is a weight. Arguing

as above this is equivalent to finding a minimal weight λ_i which is a coroot but is not a root. These are listed above.

□

Proposition 7. *For each λ_i in the table above, there is an element σ_i, of the classical Weyl group which permutes all the simple roots except α_i and and sends α_i to $-\theta_0$. This Weyl group element composed with the translation operator $l'\lambda_i$ is an isometry of C_l and Λ^l, and any two elements of Λ^l which have proportional entries in the S-matrix are related by a product of such isometries.*

Proof: If $D|l'$, the isometries of the standard Weyl alcove are discussed in [Saw02a], and shown to be of the form described in the proposition, with a set of λ_i that includes the entries in the table.

If D does not divide l', we are necessarily in the case B_n and C_n. One checks that there is a Weyl group element of the sort described in the proposition (here $\theta_0 = \phi$, and note that every root is positive or negative with respect to the resulting base). It is easy to see the inequalities defining C_l are preserved by the given transformation, and since the transformation sends weights to weights it must preserve Λ^l.

Two elements of Λ^l which have proportional entries in the S-matrix must have linearly dependent characters when applied to the image of Ψ because

$$S_{\lambda,\gamma} = \mathrm{qdim}(\lambda)\chi_\lambda(\Psi(\mathrm{qtr}_\gamma)).$$

Thus two such elements must be related by an element of $\mathcal{W}^\dagger$. Since such a transformation preserves Λ, it must be a product of an ordinary Weyl transformation and a translation generated by M and $l'\lambda_i$. Conjugating by ordinary Weyl transformations as necessary, we can write this as a product of terms of the form $T_{l'\lambda_i}\sigma_i$, times an element of $\mathcal{W}_l$, where T_λ is translation by λ. Since the element of $\mathcal{W}_l$ takes C_l to itself, it must be the identity.

□

Theorem 6. *The category $\mathcal{C}$ is modular and thus gives a $2 + 1$-dimensional TQFT except in the cases where λ_i is given in Table 1. In these cases the quotient category as described by Bruguières [Bru00] is well-defined, and gives either a modular or a spin modular category, except in the cases (with l odd) A_n with $n \equiv 1 \bmod 4$, B_n with $n \equiv 2 \bmod 4$, C_n, D_n with $n \equiv 2 \bmod 4$, and E_7 and (with l even) B_n l' odd and $n \equiv 1 \bmod 4$, and C_n with l' odd.*

Proof: Bruguières shows that a semisimple ribbon category is modular unless there is a simple object whose entries in the S-matrix are proportional to those of the trivial object. Thus by the previous proposition those without the special isometries of the Weyl alcove are modular.

In order for there to be a well-defined minimal quotient which is modular or spin modular according to Bruguières (the notion of spin-modular is introduced in [Saw02b], but the work of Bruguières extends directly to it) it suffices to show that all simple objects whose rows in the S-matrix are proportional to that of the trivial object have quantum dimension 1, are *transparent* (i.e. the square of the R-matrix acts as the identity on the tensor product of this module with any other), and the set of them forms a group under tensor product.

By the previous proposition these objects are those of the form $\iota(0)$ where ι is one of the isometries associated to the λ_i of Table 1. Since these isometries are the composition of a Weyl group element and a translation, Corollary 9 tells us $\iota(\lambda) = \lambda \hat{\otimes} \iota(0)$ (for simplicity we refer to the object indexed by $\lambda \in \Lambda^l$ as λ) so these objects form a group under tensor product (in fact the group $\mathcal{W}_l^{\Lambda}/\mathcal{W}_l$).

The square of the R matrix as a map from $W^{\lambda} \hat{\otimes} W^{\iota(0)} \cong W^{\iota(\lambda)}$ is a multiple of the identity, since this module is simple. Its quantum trace is $S_{\lambda,\iota(0)} = \mathrm{qdim}(\lambda)\,\mathrm{qdim}(\iota(0))$ and thus it is the identity. So $\iota(0)$ is transparent.

Finally a check of the quantum dimension formula (2) in each case in the table shows that the dimensions of the modules of weight $\iota(0) = l'\lambda_i - \rho + \sigma(\rho)$ is -1 in exactly the case given in the theorem, and 1 in the other cases.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark 7. *Presumably, some topological information can still be gleaned in the cases not covered by the theorem (which include $U_q(\mathfrak{su}_2)$ at odd roots of unity)*

Remark 8. *The case where l is a multiple of $2D$ corresponds to Chern-Simons theory at integer level, and is the case of the most physical interest. It also is the simplest, and indeed most of the technical details in this paper deal with the other case. The case where l is not a multiple of $2D$ corresponds to certain fractional levels, and it is not clear from the physical interpretation why we should expect modularity at these levels. The fact that it happens at all, as well as the new behavior these fractional levels exhibit (such as being defined over the alcove of the* dual *affine Weyl group, and the need in certain case to quotient to achieve modularity), are phenomena that beg an interpretation in terms of Chern-Simons theory.*

References

[AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, (220):321, 1994.

[AK92] Henning Haahr Andersen and Wen Kexin. Representations of quantum algebras. The mixed case. *J. Reine Angew. Math.*, 427:35–50, 1992.

[And92] Henning Haahr Andersen. Tensor products of quantized tilting modules. *Comm. Math. Phys.*, 149(1):149–159, 1992.

[And93] Henning Haahr Andersen. Quantum groups, invariants of 3-manifolds and semisimple tensor categories. In *Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992)*, volume 7 of *Israel Math. Conf. Proc.*, pages 1–12. Bar-Ilan Univ., Ramat Gan, 1993.

[And94] Henning Haahr Andersen. Finite-dimensional representations of quantum groups. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 1–18. Amer. Math. Soc., Providence, RI, 1994.

[And95] Henning Haahr Andersen. The irreducible characters for semi-simple algebraic groups and for quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 732–743, Basel, 1995. Birkhäuser.

[And97a] Henning Haahr Andersen. Filtrations and tilting modules. *Ann. Sci. École Norm. Sup. (4)*, 30(3):353–366, 1997.

[And97b] Henning Haahr Andersen. Modular representations of algebraic groups and relations to quantum groups. In *Algebraic and analytic methods in representation theory (Sønderborg, 1994)*, volume 17 of *Perspect. Math.*, pages 1–51. Academic Press, San Diego, CA, 1997.

[And00] Henning Haahr Andersen. A sum formula for tilting filtrations. *J. Pure Appl. Algebra*, 152(1-3):17–40, 2000. Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998).

[And01] Henning Haahr Andersen. Tilting modules for algebraic and quantum groups. In *Algebra— representation theory (Constanta, 2000)*, volume 28 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 1–21. Kluwer Acad. Publ., Dordrecht, 2001.

[AP95] Henning Haahr Andersen and Jan Paradowski. Fusion categories arising from semisimple Lie algebras. *Comm. Math. Phys.*, 169(3):563–588, 1995.

[AP00] Henning H. Andersen and Georges Papadopoulo. Liftings of quantum tilting modules. In *Representations and quantizations (Shanghai, 1998)*, pages 1–8. China High. Educ. Press, Beijing, 2000.

[APK92] Henning Haahr Andersen, Patrick Polo, and Wen Kexin. Injective modules for quantum algebras. *Amer. J. Math.*, 114(3):571–604, 1992.

[APW91] Henning Haahr Andersen, Patrick Polo, and Ke Xin Wen. Representations of quantum algebras. *Invent. Math.*, 104(1):1–59, 1991.

[APW95] H. H. Andersen, P. Polo, and K. Wen. Addendum: "Representations of quantum algebras" [Invent. Math. **104** (1991), no. 1, 1–59; MR 92e:17011]. *Invent. Math.*, 120(2):409–410, 1995.

[Bou02] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6.* Elements of Mathematics. Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley.

[Bru00] Alain Bruguières. Catégories prémodulaires, modularisations et invariants des variétés de dimension 3. *Math. Ann.*, 316(2):215–236, 2000.

[CP94] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups.* Cambridge University Press, Cambridge, 1994.

[Hum72] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag, New York, 1972.

[Jan87] Jens Carsten Jantzen. *Representations of Algebraic Groups.* Academic Press, 1987.

[Jon85] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):103–111, 1985.

[Kac83] Victor G. Kac. *Infinite Dimensional Lie Algebras.* Birkhäuser, Boston-Basel-Stuttgard, 1983.

[Kas95] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kau87] Louis H. Kauffman. State models and the Jones polynomial. *Topology*, 26(3):395–407, 1987.

[Kau90] Louis H. Kauffman. An invariant of regular isotopy. *Trans. Amer. Math. Soc.*, 318(2):417–471, 1990.

[Kir96] Alexander A. Kirillov, Jr. On an inner product in modular tensor categories. *J. Amer. Math. Soc.*, 9(4):1135–1169, 1996.

[KM91] Robion Kirby and Paul Melvin. The 3-manifold invariants of Witten and Reshetikhin-Turaev for sl(2, **C**). *Invent. Math.*, 105(3):473–545, 1991.

[KR95] Louis H. Kauffman and David E. Radford. Invariants of 3-manifolds derived from finite-dimensional Hopf algebras. *J. Knot Theory Ramifications*, 4(1):131–162, 1995.

[Kup91] Greg Kuperberg. Involutory Hopf algebras and 3-manifold invariants. *Internat. J. Math.*, 2(1):41–66, 1991.

[Le00] Thang T. Q. Le. Integrality and symmetry of quantum link invariants. *Duke Math. J.*, 102(2):273–306, 2000.

[Le03] Thang T. Q. Le. Quantum invariants of 3-manifolds: integrality, splitting, and perturbative expansion. In *Proceedings of the Pacific Institute for the Mathematical Sciences Workshop "Invariants of Three-Manifolds" (Calgary, AB, 1999)*, volume 127, pages 125–152, 2003.

[Lus88] G. Lusztig. Quantum deformations of certain simple modules over enveloping algebras. *Adv. in Math.*, 70(2):237–249, 1988.

[Lus89] George Lusztig. Modular representations and quantum groups. In *Classical groups and related topics (Beijing, 1987)*, volume 82 of *Contemp. Math.*, pages 59–77. Amer. Math. Soc., Providence, RI, 1989.

[Lus90a] George Lusztig. Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra. *J. Amer. Math. Soc.*, 3(1):257–296, 1990.

[Lus90b] George Lusztig. On quantum groups. *J. Algebra*, 131(2):466–475, 1990.

[Lus90c] George Lusztig. Quantum groups at roots of 1. *Geom. Dedicata*, 35(1-3):89–113, 1990.

[Lus92] George Lusztig. Introduction to quantized enveloping algebras. In *New developments in Lie theory and their applications (Córdoba, 1989)*, volume 105 of *Progr. Math.*, pages 49–65. Birkhäuser Boston, Boston, MA, 1992.

[Lus93] George Lusztig. *Introduction to quantum groups*, volume 110 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1993.

[Mac63] Mac Lane, Saunders. *Homology*, volume 114 of *Die Grundlehren Der Mathematischen Wissenschaften in Einzeldarstellungen*. Springer-Verlag, New York, 1963.

[Mac71] Mac Lane, Saunders. *Categories For the Working Mathematician*. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg-Berlin, 1971. ISBN 0-387-90036-5.

[Mon93] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

[MOR67] A. J. Macfarlane, L. O'Raifeartaigh, and P. S. Rao. Relationship of the internal and external multiplicity structure of compact simple Lie groups. *J. Mathematical Phys.*, 8:536–546, 1967.

[Par94] Jan Paradowski. Filtrations of modules over the quantum algebra. In *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 93–108. Amer. Math. Soc., Providence, RI, 1994.

[Ros88] Marc Rosso. Finite dimensional representations of the quantum analog of the enveloping algebra of complex simple Lie algebra. *Comm. Math. Phys.*, 117:581–593, 1988.

[Ros90] Marc Rosso. Analogues de la forme de Killing et du théorèm d'Harish-Chandra pour les groupes quantiques. *Ann. Sci. École Norm. Sup.*, 23(3):445–467, 1990.

[RT90] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990.

[RT91] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991.

[Saw02a] Stephen Sawin. Jones-Witten invariants for nonsimply connected Lie groups and the geometry of the Weyl alcove. *Adv. Math.*, 165(1):1–34, 2002.

[Saw02b] Stephen F. Sawin. Invariants of Spin three-manifolds from Chern-Simons theory and finite-dimensional Hopf algebras. *Adv. Math.*, 165(1):35–70, 2002.

[Wit89] Edward Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.

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