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Typical elements in free groups are in different doubly-twisted conjugacy classes

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Abstract

We give an easily checkable algebraic condition which implies that two elements of a finitely generated free group are members of distinct doubly-twisted conjugacy classes with respect to a pair of homomorphisms. We further show that this criterion is satisfied with probability 1 when the homomorphisms and elements are chosen at random.

1 Introduction

Let G and H be finitely generated free groups, and $\varphi, \psi : G \rightarrow H$ be homomorphisms. The group H is partitioned into the set of *doubly-twisted conjugacy classes* as follows: $u, v \in H$ are in the same class (we write $[u] = [v]$) if and only if there is some $g \in G$ with

$$u = \varphi(g)v\psi(g)^{-1}.$$

Our principal motivation for studying doubly-twisted conjugacy is Nielsen coincidence theory (see [4] for a survey), the study of the coincidence set of a pair of mappings and the minimization of this set while the mappings are changed by homotopies. Our focus on free groups is motivated specifically by the problem of computing *Nielsen classes* of coincidence points for pairs of mappings $f, g : X \rightarrow Y$, where X and Y are compact surfaces with boundary.

A necessary condition for two coincidence points to be combined by a homotopy (thus reducing the total number of coincidence points) is that they belong to the same Nielsen class. (Much of this theory is a direct generalization of similar techniques in fixed point theory, see [5].) The number of “essential” Nielsen classes is called the Nielsen number, and is a lower bound for the minimal number of coincidence points when f and g are allowed to vary by homotopies.

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In our setting, deciding when two coincidence points are in the same Nielsen class is equivalent to solving a natural doubly-twisted conjugacy problem in the fundamental groups, using the induced homomorphisms given by the pair of mappings. Thus the Nielsen classes of coincidence points correspond to twisted conjugacy classes in $\pi_1(Y)$.

The problem of computing doubly-twisted conjugacy classes in free groups is nontrivial, even in the singly-twisted case which arises in fixed point theory, where φ is an endomorphism and ψ is the identity. Existing techniques for computing doubly-twisted conjugacy are adapted from singly-twisted methods using abelian and nilpotent quotients [10]. Our main result is not adapted from singly-twisted methods: it is suited specifically for doubly-twisted conjugacy and in fact can never apply in the case where $\psi = \text{id}$.

In Section 2 we will present a remnant condition which can be used to show that two words are in different doubly-twisted conjugacy classes. In Section 3 we show in fact that this remnant condition is very common for “most” homomorphisms. In the sense of asymptotic density, we show that if the homomorphisms φ, ψ , and elements u, v are all chosen at random, then $[u] \neq [v]$ with probability 1.

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2 A remnant condition for doubly-twisted conjugacy

Given homomorphisms $\varphi, \psi : G \rightarrow H$, the *equalizer subgroup* $\text{Eq}(\varphi, \psi) \leq G$ is the subgroup

$$\text{Eq}(\varphi, \psi) = \{g \in G \mid \varphi(g) = \psi(g)\}.$$

Our first lemma is an equalizer version of a result for singly-twisted conjugacy which appears in the proof of Theorem 1.5 of [3].

Let $\langle z \rangle$ be the free group generated by z , and let $\hat{G} = G * \langle z \rangle$ and $\hat{H} = H * \langle z \rangle$, with $*$ the free product. Let $h^v = v^{-1}h v$, and $\varphi^v(g) = v^{-1}\varphi(g)v$. The following lemma holds when G and H are any groups (not necessarily free):

Lemma 1. *Given $u \in H$, let $\hat{\varphi}_u : \hat{G} \rightarrow \hat{H}$ be the extension of φ given by $\hat{\varphi}_u(z) = uzu^{-1}$, and let $\hat{\psi} : \hat{G} \rightarrow \hat{H}$ be the extension of ψ given by $\hat{\psi}(z) = z$.*

Then $[v] = [u]$ if and only if there is some $g \in G$ with $gzg^{-1} \in \text{Eq}(\hat{\varphi}_u^v, \hat{\psi})$.

Proof. First, assume that $[v] = [u]$, and let $g \in G$ be some element with

$$v = \varphi(g)u\psi(g)^{-1}.$$

Then we have

$$\begin{aligned} \hat{\varphi}_u^v(gzg^{-1}) &= v^{-1}\varphi(g)uzu^{-1}\varphi(g)^{-1}v \\ &= \psi(g)u^{-1}\varphi(g)^{-1}\varphi(g)uzu^{-1}\varphi(g)^{-1}\varphi(g)u\psi(g)^{-1} \\ &= \hat{\psi}(gzg^{-1}) \end{aligned}$$

as desired.

Now assume that there is some element $gzg^{-1} \in \text{Eq}(\hat{\varphi}_u^v, \hat{\psi})$ for $g \in G$. Consider the commutator:

$$\begin{aligned} [u^{-1}\varphi(g)^{-1}v\psi(g), z] &= u^{-1}\varphi(g)^{-1}v\psi(g)z\psi(g)^{-1}v^{-1}\varphi(g)uz^{-1} \\ &= u^{-1}\varphi(g)^{-1}v\hat{\psi}(gzg^{-1})v^{-1}\varphi(g)uz^{-1} \\ &= u^{-1}\varphi(g)^{-1}v\hat{\varphi}_u^v(gzg^{-1})v^{-1}\varphi(g)uz^{-1} \\ &= u^{-1}\varphi(g)^{-1}vv^{-1}\varphi(g)uz u^{-1}\varphi(g)^{-1}vv^{-1}\varphi(g)uz^{-1} = 1. \end{aligned}$$

Thus z commutes with $u^{-1}\varphi(g)^{-1}v\psi(g)$, and so $u^{-1}\varphi(g)^{-1}v\psi(g) = 1$, since this word does not contain the letter z . Thus $[u] = [v]$. \square

The above lemma is difficult to apply for the purpose of computing twisted conjugacy classes, since the problem of computing the equalizer subgroup of homomorphisms is difficult. In fixed point theory (where $\psi = \text{id}$), if φ is an automorphism, an algorithm of [8] is given to compute the fixed point subgroup $\text{Fix}(\varphi)$. This algorithm relies fundamentally on the methods of Bestvina and Handel [2] for representing automorphisms of free groups using train tracks, and these techniques do not extend in an obvious way to coincidence theory.

Though the equalizer subgroup is in general difficult to compute, we will show that a certain remnant property will force the equalizer subgroup to be trivial. Remnant properties were first used by Wagner in [11].

Definition 2. Let G be a finitely generated free group with a specified set of generators $\overline{G} = \{g_1, \dots, g_n\}$. The homomorphism φ has *remnant* if for each i , the word $\varphi(g_i)$ has a nontrivial subword which has no cancellation in any of the products

$$\varphi(g_j)^{\pm 1}\varphi(g_i), \quad \varphi(g_i)\varphi(g_j)^{\pm 1},$$

except for $j = i$ with exponent -1 . The maximal such noncancelling subword of $\varphi(g_i)$ is called the *remnant* of g_i , written $\text{Rem}_\varphi(g_i)$.

We will occasionally discuss the length of the remnant subwords, in one of two ways. If, for some natural number l , we have $|\text{Rem}_\varphi(g_i)| \geq l$ for all g_i , we will say that φ has *remnant length* l . For some $r \in (0, 1)$, we say that φ has *remnant ratio* r when

$$|\text{Rem}_\varphi(g_i)| \geq r|\varphi(g_i)|$$

for each i .

The condition that φ has remnant is slightly weaker than saying that $\varphi(\overline{G})$ is Nielsen reduced (see e.g. [6]), which would make some additional assumptions on the word length of the remnant subwords.

Throughout the rest of the paper, we will fix a particular generating set $\overline{G} = \{g_1, \dots, g_n\}$ for G . Given two homomorphisms $\varphi, \psi : G \rightarrow H$, there is a homomorphism $\varphi * \psi : G * G \rightarrow H$, defined as follows: Denoting $G = \langle g_1, \dots, g_n \rangle$, we write $G * G = \langle g_1, \dots, g_n, g'_1, \dots, g'_n \rangle$. Then we define $\varphi * \psi$ on the generators of $G * G$ by $\varphi * \psi(g_i) = \varphi(g_i)$ and $\varphi * \psi(g'_i) = \psi(g_i)$.

We have:

Lemma 3. *If $\varphi * \psi : G * G \rightarrow H$ has remnant, then $\varphi(G) \cap \psi(G) = \{1\}$. In particular this means that $\text{Eq}(\varphi, \psi) = \{1\}$.*

Proof. If $\varphi(G) \cap \psi(G)$ contains some nontrivial element, then we have $x, y \in G$, both nontrivial, with

$$\varphi(x)\psi(y)^{-1} = 1,$$

which is not possible when $\varphi * \psi$ has remnant: writing x and y in terms of generators will show that the left side above cannot fully cancel. \square

The two lemmas can be used to compute doubly-twisted conjugacy classes as in the following example:

Example 4. We will examine doubly-twisted conjugacy classes of the homomorphisms $\varphi, \psi : G \rightarrow G$ with $G = \langle a, b \rangle$ defined by:

$$\varphi : \begin{array}{l} a \mapsto aba \\ b \mapsto b^{-1}a \end{array} \quad \psi : \begin{array}{l} a \mapsto b^2a^{-1} \\ b \mapsto a^3 \end{array}$$

Given words $u, v \in G$, let $\eta_{(u,v)} = \hat{\varphi}_u^v * \hat{\psi}$. Our two lemmas together show that for $u, v \in G$, we will have $[u] \neq [v]$ whenever $\eta_{(u,v)} : \hat{G} * \hat{G} \rightarrow \hat{G}$ has remnant. This remnant condition can be easily checked by hand. We will apply this strategy for all words u, v of length 0 or 1.

The homomorphism $\eta = \eta_{(1,b)}$ is as follows:

$$\eta = \begin{array}{l} a \mapsto b^{-1}abab \\ b \mapsto b^{-2}ab \\ z \mapsto b^{-1}zb \\ a' \mapsto b^2a^{-1} \\ b' \mapsto a^3 \\ z' \mapsto z \end{array}$$

We can see that η has remnant, and thus by Lemma 3 that $\text{Eq}(\hat{\varphi}_1^b, \hat{\psi}) = 1$, and thus by Lemma 1 that $[1] \neq [b]$.

Checking the appropriate homomorphisms (due to the asymmetry in the role of u and v , an unsuccessful check for the pair (u, v) might actually succeed for the pair (v, u)) we obtain the following additional inequalities:

$$[a] \neq [1], \quad [a] \neq [b], \quad [a^{-1}] \neq [1], \quad [a^{-1}] \neq [b].$$

This method is somewhat tedious to perform by hand. A web-based computer implementation of the process is available for testing at the author's website.¹

The checks for remnant in the above example are equivalent to a related noncancellation condition:

¹The technique is implemented in GAP, with a web-based frontend. The front-end and GAP source code are available at <http://faculty.fairfield.edu/cstaecker>.

Theorem 5. *Let $u, v \in H$ be distinct words. If $\varphi^v * \psi$ has remnant, and if, for each generator g of $G * G$, the remnant words $\text{Rem}_{\varphi^v * \psi}(g)$ do not fully cancel in any product of the form*

$$(\varphi^v * \psi(g))v^{-1}u, \quad u^{-1}v(\varphi^v * \psi(g)), \quad (1)$$

then $[u] \neq [v]$.

Proof. Let $\hat{G} = G * \langle z \rangle$ and $\hat{H} = H * \langle z \rangle$, and let $\hat{\varphi}_u, \hat{\psi} : \hat{G} \rightarrow \hat{H}$ be defined as in Lemma 1. We will show that $\hat{\varphi}_u^v * \hat{\psi}$ has remnant.

For brevity, let $\eta = \hat{\varphi}_u^v * \hat{\psi}$, and let us denote the generators of the free product so that $\hat{G} * \hat{G} = \langle g_1, \dots, g_n, z, g'_1, \dots, g'_n, z' \rangle$. Then the homomorphism η is given by:

$$\eta = \hat{\varphi}_u^v * \hat{\psi} : \begin{array}{ll} g_i & \mapsto \varphi^v(g_i) \\ z & \mapsto v^{-1}uz u^{-1}v \\ g'_i & \mapsto \psi(g_i) \\ z' & \mapsto z \end{array}$$

To show that η has remnant, we must show that the words $\eta(g_i)$ have non-cancelling subwords in various products of the form in Definition 2. For each i , let w_i be the subword of the remnant of $\text{Rem}_{\varphi^v * \psi}(g_i)$ which has no cancellation in any of the products in (1). Similarly let w'_i be the subword of the remnant of $\text{Rem}_{\varphi^v * \psi}(g'_i)$ with no cancellation in any of the products in (1).

Let us first examine subwords of $\eta(g_i)$ in products of the form

$$\eta(g_i)\eta(g_j)^{\pm 1} = \varphi^v(g_i)\varphi^v(g_j)^{\pm 1} \quad \text{or} \quad \eta(g_i)\eta(g'_j)^{\pm 1} = \varphi^v(g_i)\psi(g_j)^{\pm 1}.$$

In these products, w_i will remain uncanceled (unless $j = i$ with exponent -1) because it is a subword of $\text{Rem}_{\varphi^v * \psi}(g_i)$.

Now consider

$$\eta(g_i)\eta(z)^{\pm 1} = \varphi^v(g_i)v^{-1}uz^{\pm 1}u^{-1}v.$$

Here w_i will remain uncanceled by the hypotheses on products of the form in (1), together with the fact that no cancellation can occur with the $z^{\pm 1}$ because u and v do not use the letter z . Finally we must consider products of the form

$$\eta(g_i)\eta(z')^{\pm 1} = \varphi^v(g_i)z,$$

in which clearly w_i does not cancel.

We have shown that w_i has no cancellation in products of the form in Definition 2 involving $\eta(g_i)$ on the left. Similar arguments will show that w_i has no cancellation in products involving $\eta(g_i)$ on the right. Identical arguments will show that the words w'_i are uncanceled in various products involving $\eta(g'_i)$, and thus $\hat{\varphi}_u^v * \hat{\psi}$ has remnant. Since $\hat{\varphi}_u^v * \hat{\psi}$ has remnant, we have $\text{Eq}(\hat{\varphi}_u^v, \hat{\psi}) = \{1\}$ by Lemma 3, and thus $[u] \neq [v]$ by Lemma 1. \square

Note that Theorem 5 cannot be used in fixed point theory to distinguish singly-twisted conjugacy classes, since $\varphi^v * \text{id}$ can never have remnant.

3 Generic properties

A theorem of Robert F. Brown in [11] shows that “most” homomorphisms have remnant. Theorem 3.7 of that paper is:

Lemma 6. *Let G be a free group with generators g_1, \dots, g_n with $n > 1$. Given any $\epsilon > 0$, there exists some $M > 0$ such that, if $\varphi : G \rightarrow G$ is an endomorphism chosen at random with $|\varphi(g_i)| \leq M$ for all generators $g_i \in G$, then the probability that φ has remnant is greater than $1 - \epsilon$.*

The above is the only result of its kind typically referenced in the Nielsen theory literature, but it is in the spirit of a well established theory of generic group properties. (See [9] for a survey.)

For a free group G and a natural number p , let G_p be the subset of all words of length at most p . For a subset $S \subset G$, let $S_p = S \cap G_p$. The *asymptotic density* (or simply *density*) of S is defined as

$$D(S) = \lim_{p \rightarrow \infty} \frac{|S_p|}{|G_p|},$$

where $|\cdot|$ denotes the cardinality. The set S is said to be *generic* if $D(S) = 1$.

Similarly, if $S \subset G^l$ is a set of l -tuples of elements of G , the asymptotic density of S is defined as

$$D(S) = \lim_{p \rightarrow \infty} \frac{|S_p|}{|(G_p)^l|},$$

where $S_p = S \cap (G_p)^l$, and S is called *generic* if $D(S) = 1$.

A homomorphism on the free group $G = \langle g_1, \dots, g_n \rangle$ is equivalent combinatorially to an n -tuple of elements of G (the n elements are the words $\varphi(g_i)$ for each generator g_i). Thus the asymptotic density of a set of homomorphisms can be defined in the same sense as above, viewing the set of homomorphisms as a collection of n -tuples. The statement of Lemma 6, then, is simply that the set of endomorphisms $G \rightarrow G$ with remnant is generic. Similarly we can define the density of a set of pairs of homomorphisms by viewing it as a collection of $2n$ -tuples (a pair of homomorphisms is equivalent to a pair of n -tuples).

The statement of Lemma 6 can be strengthened and extended easily to general homomorphisms (possibly non-endomorphisms) using a generic property from [1]. Consider the setting of homomorphisms $G \rightarrow H$, where G and H are finitely generated free and H has more than one generator. Lemma 3 of [1] implies that for any l , the collection of l -tuples of H which are Nielsen reduced (when viewed as sets of elements of H) is generic. This directly gives

Lemma 7. *If the rank of H is greater than 1, then the set of homomorphisms $G \rightarrow H$ with remnant is generic.*

Applying the above to homomorphisms $G * G \rightarrow H$ and applying Lemma 3 gives an interesting corollary:

Corollary 8. *If the rank of H is greater than 1, then the set of pairs of homomorphisms $\varphi, \psi : G \rightarrow H$ with $\text{Eq}(\varphi, \psi) = \{1\}$ is generic.*

This gives a somewhat counterintuitive result: If F_i is the free group of rank i , a pair of homomorphisms from F_{1000} to F_2 will generically have images whose intersection is trivial. (See results of a similar spirit in [7], e.g. that homomorphisms of free groups are generically injective but not surjective.)

The cited result from [1] is quite a bit stronger: it is shown that the set of subsets of G having small cancellation property $C'(\lambda)$ is generic for any $\lambda > 0$. This gives stronger results concerning remnant properties of generic homomorphisms:

Lemma 9. *Let the rank of H be greater than 1. Then:*

- *For any natural number l , the set of homomorphisms $\varphi : G \rightarrow H$ with remnant length l is generic.*
- *For any $r \in (0, 1)$, the set of homomorphisms $\varphi : G \rightarrow H$ with remnant ratio r is generic.*

We include the above lemma for the sake of completeness, but we will not actually require its full strength in order to prove our generic property for doubly-twisted conjugacy.

Theorem 10. *Let G and H be free groups, with the rank of H greater than 1. Then the set*

$$S = \{(\varphi, \psi, u, v) \mid [u] \neq [v]\}$$

is generic.

Proof. We will slightly extend our free-product notation for homomorphisms as follows: for a homomorphism $\varphi : G \rightarrow H$ and a word $w \in H$, let $\langle x \rangle$ be the free group generated by some new letter x . Then define $\varphi * w : G * \langle x \rangle \rightarrow H$ by $\varphi * w(g_i) = \varphi(g_i)$ for g_i a generator of G , and $\varphi * w(x) = w$.

By Theorem 5, S contains all tuples (φ, ψ, u, v) such that $\varphi^v * \psi * uv^{-1}$ has remnant. This remnant condition will be satisfied when the bracketed words below have subwords which do not cancel in any of the following products (except those which are trivial):

$$\begin{array}{lll} [\varphi^v(g_i)](\varphi^v(g_j))^{\pm 1}, & [\varphi^v(g_i)]\psi(g_j)^{\pm 1}, & [\varphi^v(g_i)](uv^{-1})^{\pm 1}, \\ (\varphi^v(g_j))^{\pm 1}[\varphi^v(g_i)], & \psi(g_j)^{\pm 1}[\varphi^v(g_i)], & (uv^{-1})^{\pm 1}[\varphi^v(g_i)], \\ [\psi(g_i)](\varphi^v(g_j))^{\pm 1}, & [\psi(g_i)]\psi(g_j)^{\pm 1}, & [\psi(g_i)](uv^{-1})^{\pm 1}, \\ (\varphi^v(g_j))^{\pm 1}[\psi(g_i)], & \psi(g_j)^{\pm 1}[\psi(g_i)], & (uv^{-1})^{\pm 1}[\psi(g_i)], \\ [uv^{-1}](\varphi^v(g_j))^{\pm 1}, & [uv^{-1}]\psi(g_j)^{\pm 1}. & \end{array}$$

It can be verified that there will be noncanceling subwords in the bracketed parts above when $\varphi * \psi * u * v$ has remnant. We will verify the first and last of these:

In the product

$$[\varphi^v(g_i)](\varphi^v(g_j))^{\pm 1} = [v^{-1}\varphi(g_i)v]v^{-1}\varphi(g_j)^{\pm 1}v,$$

if $\varphi * \psi * u * v$ has remnant, then a portion of $\varphi(g_i)$ will remain uncanceled. Now consider the product:

$$[uv^{-1}]\psi(g_j)^{\pm 1}.$$

Again, if $\varphi * \psi * u * v$ has remnant, then a portion of u will remain uncanceled. It is easy to check that various remnant words of $\varphi * \psi * u * v$ are similarly uncanceled in the other of the 14 products above.

Thus S contains the set of tuples (φ, ψ, u, v) such that $\varphi * \psi * u * v$ has remnant. But this set of tuples is generic by Lemma 9, since a choice of a tuple with $\varphi * \psi * u * v$ having remnant is combinatorially equivalent to choosing a single homomorphism $\eta : F_{2n+2} \rightarrow H$, where F_{2n+2} is the free group on $2n + 2$ generators. Since S contains a generic set, it is itself generic. \square

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