1-1-2011

Spectral analysis of the transfer operator for the Lorentz gas

Mark Demers
Fairfield University, mdemers@fairfield.edu

Hong-Kun Zhang

Copyright 2011 American Institute of Mathematical Sciences, Journal of Modern Dynamics.

Peer Reviewed

Repository Citation
Demers, Mark and Zhang, Hong-Kun, "Spectral analysis of the transfer operator for the Lorentz gas" (2011). Mathematics Faculty Publications. 34.
http://digitalcommons.fairfield.edu/mathandcomputerscience-facultypubs/34

Published Citation

This Article is brought to you for free and open access by the Mathematics Department at DigitalCommons@Fairfield. It has been accepted for inclusion in Mathematics Faculty Publications by an authorized administrator of DigitalCommons@Fairfield. For more information, please contact digitalcommons@fairfield.edu.
SPECTRAL ANALYSIS OF THE TRANSFER OPERATOR
FOR THE LORENTZ GAS

MARK F. DEMERS AND HONG-KUN ZHANG
(Communicated by Dmitry Dolgopyat)

Abstract. We study the billiard map associated with both the finite- and
infinite-horizon Lorentz gases having smooth scatterers with strictly positive
curvature. We introduce generalized function spaces (Banach spaces of dis-
tributions) on which the transfer operator is quasicompact. The mixing prop-
erties of the billiard map then imply the existence of a spectral gap and re-
lated statistical properties such as exponential decay of correlations and the
Central Limit Theorem. Finer statistical properties of the map such as the
identification of Ruelle resonances, large deviation estimates and an almost-
sure invariance principle follow immediately once the spectral picture is es-
tablished.

1. Introduction

Much attention has been given in recent years to developing a framework
to study directly the transfer operator associated with hyperbolic maps on an
appropriate Banach space. The goal of such a functional analytic approach is
first to use the smoothing properties of the transfer operator to prove its quasi-
compactness and then to derive statistical information about the map from the
peripheral spectrum. For expositions of this subject, see [1, 23, 29].

The link between the transfer operator and the statistical properties of the
map traces back to classical results regarding Markov chains [20, 24, 33]. In
the context of deterministic systems, this approach was first adapted to over-
come the problem of discontinuities for expanding maps by using the smooth-
ing effect of the transfer operator on functions of bounded variation [28, 26,
42, 10, 43, 44, 11]. Its extension to hyperbolic maps followed, using simultane-
ously the smoothing properties of the transfer operator in unstable directions
and the contraction present in the stable directions: first to Anosov diffeomor-
phisms [39, 40, 41, 9, 2, 5, 21] and more recently to piecewise hyperbolic maps
[18, 3, 4]. Two crucial assumptions in the treatment of the piecewise hyperbolic
case in two dimensions have been: (1) the map has a finite number of singularity curves and (2) the map admits a smooth extension up to the closure of each of its domains of definition. These assumptions and other technical difficulties have thus far prevented this approach from being successfully carried out for dispersing billiards.

In this paper, we apply the functional analytic approach to the billiard map associated with both a finite- and infinite-horizon Lorentz gas having smooth scatterers with strictly positive curvature. We introduce generalized function spaces (Banach spaces of distributions) on which the transfer operator is quasi-compact. The mixing properties of the billiard map then imply the existence of a spectral gap, the exponential decay of correlations and finer statistical properties such as Ruelle resonances. Many limit theorems such as local large deviation estimates, a Central Limit Theorem, and an almost sure invariance principle for both invariant and noninvariant measures also follow immediately once the spectral picture is established.

Although the exponential decay of correlations and many limit theorems are already known for such classes of billiards [45, 13, 36, 32], the present approach provides a unified and greatly simplified framework in which to achieve these results and completely bypasses previous methods which relied on constructing countable Markov partitions [7, 8], Markov extensions [45, 13, 16], or magnets for coupling arguments [14], all of which require a deep understanding of the regularity properties of the foliations. Indeed, we avoid entirely the need to work with the holonomy map matching unstable curves along real stable manifolds, which is a major technical difficulty present in each of the previous approaches.

In addition, the current functional analytic framework allows immediate extensions of well-known limit theorems to noninvariant measures. For example, we prove in Theorem 2.6 that our large deviation rate function is independent of the probability measure in our Banach space with which we measure the asymptotic deviations (see Section 2.4). Although the limit theorems with respect to invariant measures presented in Theorem 2.6 are already known for this class of billiards, the extensions to noninvariant measures constitute new results, with the partial exception of [19], which dealt with large deviations only. Finally, the spectral picture obtained via the method in this paper has been shown to be robust under a wide variety of perturbations in a number of settings [6, 27] (see also the treatment of perturbations using norms similar to those in this paper in [18]), and it is expected that the present framework will allow the unified treatment of large classes of perturbations in a way previously unattainable for billiards.

The paper is organized as follows. In Section 2, we define the Banach spaces on which we will study the transfer operator and state our main results. The norms we define follow closely those introduced in [18], with the addition of an extra weighting factor to counteract the blowup of the Jacobian of the map near singularities. In order to control distortion, we introduce additional cuts
at the boundaries of homogeneity strips which implies that our expanded singularity sets comprise a countably infinite number of curves in both the finite- and infinite-horizon cases. In Section 3, we prove the necessary growth lemmas to control the cutting generated by the expanded singularity sets and prove preliminary properties of our Banach spaces including embeddings and compactness. Section 4 contains the required Lasota–Yorke inequalities and in Section 5 we characterize the peripheral spectrum and prove some related statistical properties. Section 6 contains the proofs of the limit theorems mentioned above.

2. Setting, Definitions and Results

2.1. Billiard maps associated with a Lorentz gas. We define here the class of maps to which our results apply and take the opportunity to establish some notation. Let \( \{\Gamma_i\}_{i=1}^{d} \) be pairwise disjoint, simply connected convex regions in \( \mathbb{T}^2 \) having \( \mathcal{C}^3 \) boundary curves \( \partial\Gamma_i \) with strictly positive curvature. We consider the billiard flow on the table \( Q = \mathbb{T}^2 \setminus \bigcup_i [\text{interior } \Gamma_i] \) induced by a particle traveling at unit speed and undergoing elastic collisions at the boundaries. The phase space for the billiard flow is \( \mathcal{M} = Q \times S^1 / \sim \) with the conventional identifications at the boundaries. Define \( M = \bigcup_i \partial\Gamma_i \times [-\pi/2, \pi/2] \). The billiard map \( T: M \to M \) is the Poincaré map corresponding to collisions with the scatterers. We will denote coordinates on \( M \) by \( (r, \varphi) \), where \( r \in \bigcup_i \partial\Gamma_i \) is parametrized by arclength and \( \varphi \) is the angle that the unit tangent vector at \( r \) makes with the normal pointing into the domain \( Q \). \( T \) preserves a probability measure \( \mu \) defined by \( d\mu = c \cos\varphi \, dr \, d\varphi \) on \( M \), where \( c \) is the normalizing constant.

For any \( x = (r, \varphi) \in M \), we denote by \( \tau(x) \) the time of the first (nontangential) collision of the trajectory starting at \( x \) under the billiard flow. The billiard map \( T \) is defined whenever \( \tau(x) < \infty \) and is known to be uniformly hyperbolic, although its derivative \( DT \) becomes infinite near singularities (see for example \([15, \text{Chapter 4}]\)). We say \( T \) has \textit{finite horizon} if there is an upper bound on the function \( \tau \). Otherwise, we say \( T \) has \textit{infinite horizon}.

2.2. Transfer Operator. We define scales of spaces using a set of \textit{admissible curves} \( \mathcal{W}^s \) (defined in Section 3.1) on which we define the action of the \textit{transfer operator} \( \mathcal{L} \) associated with \( T \). Such curves are homogeneous stable curves whose length is smaller than some fixed \( \delta_0 \). Define \( T^{-n}\mathcal{W}^s \) to be the set of homogeneous stable curves \( W \) such that \( T^n \) is smooth on \( W \) and \( T^i W \in \mathcal{W}^s \) for \( 0 \leq i \leq n \). It follows from the definition that \( T^{-n}\mathcal{W}^s \subset \mathcal{W}^s \).

We denote (normalized) Lebesgue measure on \( M \) by \( m \), i.e., \( dm = c\, dr \, d\varphi \). For \( W \in T^{-n}\mathcal{W}^s \), a complex-valued test function \( \psi: M \to \mathbb{C} \) and \( 0 < p \leq 1 \), define \( H^p_W(\psi) \) to be the Hölder constant of \( \psi \) on \( W \) with exponent \( p \) measured in the Euclidean metric. Define \( H^p_W(\psi) = \sup_{W \in T^{-n}\mathcal{W}^s} H^p_W(\psi) \) and let

\[
\hat{\mathcal{C}}^p(T^{-n}\mathcal{W}^s) = \{ \psi: M \to \mathbb{C} \mid |\psi|_\infty + H^p_W(\psi) < \infty \}
\]
denote the set of complex-valued functions which are Hölder-continuous on elements of $T^{-n}\mathcal{W}_s$. The set $\mathcal{C}^p(T^{-n}\mathcal{W}_s)$ with the norm

$$||\psi||_{\mathcal{C}^p(T^{-n}\mathcal{W}_s)} = ||\psi||_\infty + H^p_0(\psi)$$

is a Banach space. We define $\mathcal{C}^p(T^{-n}\mathcal{W}_s)$ to be the closure of $\mathcal{C}^1(T^{-n}\mathcal{W}_s)$ in $\mathcal{C}^p(T^{-n}\mathcal{W}_s)$. Similarly, we define $\mathcal{C}^p(T^n\mathcal{W}_u)$ and $\mathcal{C}^p(T^n\mathcal{W}_u)$ for each $n \geq 0$, the set of functions which are Hölder-continuous with exponent $p$ on unstable curves in $T^n\mathcal{W}_u$, defined in Section 3.1.

It follows from (4.3) that if $\psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}_s)$, then $\psi \circ T \in \mathcal{C}^p(T^{-n}\mathcal{W}_s)$. Similarly, if $\xi \in \mathcal{C}^1(T^{-(n-1)}\mathcal{W}_s)$, then $\xi \circ T \in \mathcal{C}^1(T^{-n}\mathcal{W}_s)$. These two facts together imply that if $\psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}_s)$, then $\psi \circ T \in \mathcal{C}^p(T^{-n}\mathcal{W}_s)$.

If $h \in \mathcal{C}^p(T^{-n}\mathcal{W}_s)'$, is an element of the dual of $\mathcal{C}^p(T^{-n}\mathcal{W}_s)$, then $\mathcal{L} : \mathcal{C}^p(T^{-n}\mathcal{W}_s)' \rightarrow \mathcal{C}^p(T^{-(n-1)}\mathcal{W}_s)'$ acts on $h$ by

$$\mathcal{L} h(\psi) = h(\psi \circ T) \quad \forall \psi \in \mathcal{C}^p(T^{-(n-1)}\mathcal{W}_s).$$

If $h \in L^1(M, m)$, then $h$ is canonically identified with a signed measure absolutely continuous with respect to Lebesgue, which we shall also call $h$, i.e.,

$$h(\psi) = \int_M \psi h \, dm.$$

With the above identification, we write $L^1(M, m) \subset (\mathcal{C}^p(T^{-n}\mathcal{W}_s)')$ for each $n \in \mathbb{N}$. Then restricted to $L^1(M, m)$, $\mathcal{L}$ acts according to the familiar expression

$$\mathcal{L}^n h = h \circ T^{-n} |DT^n(T^{-n})|^{-1}$$

for any $n \geq 0$ and any $h \in L^1(M, m)$, where $|DT^n|$ denotes $|\det DT^n|$ to simplify notation.

2.3. Definition of the Norms. The norms we introduce below are defined via integration on the set of admissible stable curves $\mathcal{W}_s$ referred to in Section 2.2. In Section 3.1 we define precisely the notion of a distance $d_{\mathcal{W}_s}(\cdot, \cdot)$ between such curves as well as a distance $d_q(\cdot, \cdot)$ defined among functions supported on these curves.

The motivation for these norms is the following: We expect the action of the transfer operator to increase regularity in the unstable direction and to decrease regularity in the stable direction, so we integrate along stable curves in order to average the action of the transfer operator in the stable direction. The unstable norm $\| \cdot \|_u$ morally measures a Hölder constant in the unstable direction by comparing the norms of an element of $\mathcal{B}$ on two stable curves lying close together. The weights $\cos W$ assigned to the test functions are introduced to counteract the blowup of the Jacobian near singularities; they also help us sum over homogeneity strips as in the proof of Lemma 3.9. The weight $\alpha$ in $\| \cdot \|_s$ is important for the proof of compactness (see Lemma 3.10) as well as the Lasota–Yorke estimate for $\| \cdot \|_u$ (see Section 4.3).

Given a curve $W \in \mathcal{W}_s$, we denote by $m_W$ the unnormalized Lebesgue (arc-length) measure on $W$. We set $|W| = m_W(W)$. We also denote the Euclidean
metric on $W$ by $d_W(\cdot, \cdot)$. With a slight abuse of notation, we define $\cos W$ to be the average value of $\cos \varphi$ on $W \in \mathcal{W}$, i.e., $\cos W = |W|^{-1} \int_W \cos \varphi \, dm_{W}$.

For $0 \leq p \leq 1$, denote by $\mathcal{C}^p(W)$ the set of continuous complex-valued functions on $W$ with Hölder exponent $p$, measured in the Euclidean metric. We then denote by $\mathcal{C}^p(W)$ the closure of $\mathcal{C}^1(W)$ in the $\mathcal{C}^p$-norm\footnote{Note that for $p < 1$, while $\mathcal{C}^p(W)$ may not contain all of $\mathcal{C}^p(W)$, it does contain $\mathcal{C}^{p'}(W)$ for all $p' > p$.}: $|\psi|_{\mathcal{C}^p(W)} = |\psi|_{\mathcal{C}^1(W)} + H_W^p(\psi)$, where $H_W^p(\psi)$ is the Hölder constant of $\psi$ along $W$. Note that with this definition, $|\psi_1\psi_2|_{\mathcal{C}^p(W)} \leq |\psi_1|_{\mathcal{C}^p(W)}|\psi_2|_{\mathcal{C}^p(W)}$. We define $\mathcal{C}^p(M)$ and $\mathcal{C}^p(M)$ similarly.

For $\alpha \geq 0$, define the following norms for test functions,

$$|\psi|_{W, \alpha, p} := |W|^\alpha \cdot \cos W \cdot |\psi|_{\mathcal{C}^p(W)}.$$  

Now fix $0 < p \leq 1/3$. Given a function $h \in \mathcal{C}^1(M)$, define the weak norm of $h$ by

$$|h|_W := \sup_{W \in \mathcal{W}} \sup_{\psi \in \mathcal{C}^p(W)} \int_W h\psi \, dm_W.$$  

Choose\footnote{The restrictions on the constants are placed according to the dynamical properties of $T$. For example, $p \leq 1/3$ due to the distortion estimate (3.1) while $a < 1/6$ so that Lemma 3.4 can be applied with $\zeta = 1 - \alpha > 5/6$.} $\alpha, \beta, q > 0$ such that $\alpha < 1/6$, $q < p$ and $\beta \leq \min\{\alpha, p - q\}$. We define the strong stable norm of $h$ as

$$\|h\|_{s} := \sup_{W \in \mathcal{W}} \sup_{\psi \in \mathcal{C}^q(W)} \frac{1}{\|\psi\|_{W, \alpha, q}^\beta} \left( \int_{W_1} |h\psi_1| \, dm_W - \int_{W_2} |h\psi_2| \, dm_W \right),$$  

and the strong unstable norm as

$$\|h\|_{u} := \sup_{\varepsilon \leq \delta_0} \sup_{W_1, W_2 \in \mathcal{W}^s} \sup_{\psi_1, \psi_2 \in \mathcal{C}^p(W_1)} \frac{1}{\varepsilon^{\beta}} \left( \int_{W_1} |h\psi_1| \, dm_W - \int_{W_2} |h\psi_2| \, dm_W \right),$$  

where $\varepsilon > 0$ is chosen less than $\delta_0$, the maximum length of $W \in \mathcal{W}^s$ which is determined after (3.2). We then define the strong norm of $h$ by

$$\|h\|_{\mathcal{B}} = \|h\|_{s} + b\|h\|_{u},$$  

where $b$ is a small constant chosen in Section 2.4.

We define $\mathcal{B}$ to be the completion of $\mathcal{C}^1(M)$ in the strong norm and $\mathcal{B}_w$ to be the completion of $\mathcal{C}^1(M)$ in the weak norm. In Section 4, we will actually apply these norms to functions of the form $\mathcal{L}h$ where $h \in \mathcal{C}^1(M)$. The fact that $\mathcal{L}h \in \mathcal{B}$ when $h \in \mathcal{C}^1(M)$ is established in Lemma 3.8.
2.4. Statement of Results. We assume throughout that $T$ is the billiard map associated to a finite- or infinite-horizon Lorentz gas as described in Section 2.1.

The first result gives a more concrete description of the above abstract spaces.

**Lemma 2.1.** For $\gamma > 2\beta$ and each $n \geq 0$, $\mathcal{C}^\gamma(M) \hookrightarrow \mathcal{B} \hookrightarrow \mathcal{B}_w \hookrightarrow (\mathcal{C}^p(T^{-n}W^s))'$, each of the embeddings is continuous and the first two are injective. Moreover, $\mathcal{L}$ is well defined as an operator on both $\mathcal{B}$ and $\mathcal{B}_w$.

**Proof.** The continuity of the embeddings follows from the fact that $\|h\|_{s} \leq C|h|_{\mathcal{C}^\gamma(M)}$ by (3.24) in the proof of Lemma 3.7, that $\|\cdot\|_{w} \leq \|\cdot\|_{s}$ by definition, and Lemma 3.9 which implies that $|h(\psi)| \leq C|h|_{w}|\psi|_{\mathcal{C}^p(T^{-n}W^s)}$ for all $h \in \mathcal{B}_w$ and any $\psi \in \mathcal{C}^p(T^{-n}W^s)$.

The injectivity of the first embedding is immediate while that of the second follows from the fact that our test functions for $\|\cdot\|_{s}$ are in $\mathcal{C}^q(M)$ rather than $\mathcal{C}^{q}(M)$. Finally, the fact that $\mathcal{L}$ is well defined on $\mathcal{B}$ follows from Section 3.8. The proof that $\mathcal{L}$ is well defined on $\mathcal{B}_w$ is similar and is omitted. \qed

**Remark 2.2.** One could make the embedding $\mathcal{B}_w \hookrightarrow (\mathcal{C}^p(T^{-n}W^s))'$ injective by using test functions $\psi$ in the weak norm satisfying $|\psi|_{\mathcal{C}^p(W)}|W|^a \cos W \leq 1$, with the requirements that $p < a < \alpha$ and $\beta \leq \alpha - a$. We do not do this since we do not need the injectivity of this embedding for any of the results of our paper. Also the modification would complicate the Lasota–Yorke inequalities slightly and would reduce our best estimate on the essential spectral radius to be $\Lambda^{-1/12}$ (see Remark 2.4).

The following inequalities are proven in Section 4.

**Proposition 2.3.** Let $\Lambda > 1$ be the minimum expansion from (2.8) and let $\delta_1 > 0$, $\theta_1 < 1$ be constants defined by (3.3). There exists $C > 0$ such that for all $h \in \mathcal{B}$ and $n \geq 0$,

\begin{align}
|\mathcal{L}^n h|_w & \leq C|h|_w, \\
\|\mathcal{L}^n h\|_{s} & \leq C(\theta_1^{1-a} \Lambda^n + \Lambda^{-\alpha n})\|h\|_{s} + C\delta_1^{-a}|h|_w, \\
\|\mathcal{L}^n h\|_{u} & \leq Cn^\beta \Lambda^{-\beta n}\|h\|_{u} + CC_1^n\|h\|_{s},
\end{align}

where $C_1 > 0$ is from Lemma 3.4

If we choose $1 > \sigma > \max(\Lambda^{-\beta}, \theta_1^{-1}, \Lambda^{-q})$, then there exists $N \geq 0$ such that

\begin{align}
\|\mathcal{L}^N h\|_{s} & = \|\mathcal{L}^N h\|_{s} + b\|\mathcal{L}^N h\|_{u} \\
\leq \frac{\sigma^N}{2} \|h\|_{s} + C\delta_1^{-a}|h|_w + b\sigma^N\|h\|_{u} + bCC_1^N\|h\|_{s} \\
\leq \sigma^N \|h\|_{s} + C\delta_1|h|_w,
\end{align}

provided $b$ is chosen small enough with respect to $N$. The above represents the traditional Lasota–Yorke inequality.

The final ingredient in the strategy to prove the quasicompactness of the operator $\mathcal{L}$ is the relative compactness of the unit ball of $\mathcal{B}$ in $\mathcal{B}_w$. This is proven
in Lemma 3.10. It thus follows by standard arguments (see [1, 23]) that the essential spectral radius of $\mathcal{L}$ on $B$ is bounded by $\sigma$, while the estimate for the spectral radius is one.

**Remark 2.4.** Since by (3.3) we choose $\theta_1 \leq \Lambda^{-1/2}$, and given the constraints among $\beta$, $\alpha$ and $q$, our best estimate on the essential spectral radius is $\Lambda^{-1/6}$.

With these estimates on the spectral radius and essential spectral radius of $\mathcal{L}$, we next prove the spectral decomposition of the transfer operator in Section 5. Those results and the resulting information about the statistical properties of $T$ are summarized in the following theorem. We denote by $\Pi_0$ the projection onto the eigenspace of $\mathcal{L}$ corresponding to the eigenvalue 1.

**Theorem 2.5.** The peripheral spectrum of $\mathcal{L}$ on $B$ consists of a simple eigenvalue at 1. The unique (normalized) eigenvector corresponding to 1 is the smooth invariant measure $d\mu = \rho \, dm$, where $\rho = c \cos \varphi$ and $c$ is a normalizing constant. In addition:

1. For any probability measure $\nu \in B$, we have $\lim_{n \to \infty} \|\mathcal{L}^n \nu - \mu\|_{B} = 0$ and this convergence occurs at an exponential rate given by $\sigma' :=$ the spectral radius of $\mathcal{L} - \Pi_0$ on $B$, $\sigma' < 1$.
2. $(T, \mu)$ exhibits exponential decay of correlations for Hölder observables. More precisely, for $\phi \in \mathcal{C}^\gamma(M)$, $\gamma > 2\beta$, and $\psi \in \mathcal{C}^p(W^s)$, we have
$$\left| \int_M \phi \psi \circ T^n \, d\mu - \int \phi \, d\mu \int \psi \, d\mu \right| \leq C(\sigma')^n |\phi|_{\mathcal{C}^\gamma(M)} |\psi|_{\mathcal{C}^p(W^s)}.$$  
3. More generally, the Fourier transform of the correlation function (sometimes called the power spectrum) admits a meromorphic extension in the annulus $\{z \in \mathbb{C}; \sigma < |z| < \sigma^{-1}\}$ and the poles (Ruelle resonances) correspond exactly to the eigenvalues of $\mathcal{L}$, where $\sigma$ is from (2.7).

Item (1) is proved in Section 5.1 while items (2) and (3) are proved in Section 5.2.

2.4.1. Limits theorems for billiards. Once the spectral picture described above has been established, a variety of limit theorems become immediately accessible, testifying to the concise nature of the present approach. Such limit theorems have been the subject of many recent studies and we refer the interested reader to the following partial list [23, 31, 12, 36, 22].

We state several limit theorems here and show how they follow from our functional analytic framework in Section 6. Although these limit theorems with respect to invariant measures are known for this class of billiards, their extension to noninvariant probability measures is a new result, with the exception of [19]. Throughout this section, $g$ denotes a real-valued function in $\mathcal{C}^\gamma(M)$, where $\gamma = \max\{p, 2\beta + \varepsilon\}$ for some $\varepsilon > 0$, and $S_n g = \sum_{j=0}^{n-1} g \circ T^j$. 

**Remarks:****
Large deviation estimates. Large deviation estimates give exponential bounds on the rate of convergence of $\frac{1}{n}S_n g$ to $\mu(g)$. They typically take the form

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mu \left( x \in M : \frac{1}{n} S_n g(x) \in [t - \varepsilon, t + \varepsilon] \right) = -I(t),$$

where $I(t) \geq 0$ is called the rate function. More generally, one can ask about the above limit when $\mu$ is replaced by a noninvariant measure, for example Lebesgue measure. In the present context, we prove a large deviation estimate for all probability measures in $\mathcal{B}$ with the same rate function $I$.

Central Limit Theorem. Assume $\mu(g) = 0$ and let $(g \circ T^j)_{j \in \mathbb{N}}$ be a sequence of random variables on the probability space $(M, \nu)$, where $\nu$ is a (not necessarily invariant) probability measure on the Borel $\sigma$-algebra. We say that the triple $(g, T, \nu)$ satisfies a Central Limit Theorem if there exists a constant $\varsigma^2 \geq 0$ such that

$$S_n g \dist \mathcal{N}(0, \varsigma^2),$$

where $\mathcal{N}(0, \varsigma^2)$ denotes the normal distribution with mean 0 and variance $\varsigma^2$.

Almost-sure Invariance Principle. Assume again that $\mu(g) = 0$ and as above distribute $(g \circ T^j)_{j \in \mathbb{N}}$ according to a probability measure $\nu$. Suppose there exists $\varepsilon > 0$, a probability space $\Omega$ with a sequence of random variables $\{X_n\}$ satisfying $S_n g \dist X_n$ and a Brownian motion $W$ with variance $\varsigma^2 \geq 0$ such that

$$X_n = W(n) + O(n^{1/2-\varepsilon}) \quad \text{as } n \to \infty \text{ almost-surely in } \Omega.$$

Then we say that the process $(g \circ T^j)_{j \in \mathbb{N}}$ satisfies an almost-sure invariance principle.

**Theorem 2.6.** Let $\gamma = \max\{p, 2\beta + \varepsilon\}$, for some $\varepsilon > 0$. If $g \in \mathcal{C}^\gamma(M)$, then

(a) $g$ satisfies a large deviation estimate with uniform rate function $I$ for all (not necessarily invariant) probability measures $\nu \in \mathcal{B}$.

Assume that $\mu(g) = 0$, let $\nu \in \mathcal{B}$ be a probability measure and distribute $(g \circ T^j)_{j \in \mathbb{N}}$ according to $\nu$. Then,

(b) $(g, T, \nu)$ satisfies the Central Limit Theorem;

(c) the process $(g \circ T^j)_{j \in \mathbb{N}}$ satisfies an almost-sure invariance principle.

The proof of this theorem appears in Section 6.

2.5. Known Facts about the Lorentz gas. Before exploring the properties of the Banach spaces we have defined, we recall some of the important properties of dispersing billiards that we shall need and refer the reader to [7, 8, 15] for details.
2.5.1. **Hyperbolicity.** Since we have assumed that the scatterers have strictly positive curvature $\mathcal{K}(x)$ at each $x \in M$, there exist constants $\mathcal{K}_{\text{min}}, \mathcal{K}_{\text{max}}, \tau_{\text{min}}$ such that

\[ 0 < \mathcal{K}_{\text{min}} \leq \mathcal{K}(x) \leq \mathcal{K}_{\text{max}}, \quad \tau_{\text{min}} \leq \tau(x), \quad \forall x \in M.\]

This allows us to define global stable and unstable cones as follows. Let $(dr, d\varphi)$ be an element of the tangent space. Then

\[ C^u(x) := \{(dr, d\varphi) \in \mathcal{T}_x M : \mathcal{K}_{\text{min}} \leq \frac{d\varphi}{dr} \leq \mathcal{K}_{\text{max}} + \frac{1}{\tau_{\text{min}}} \} \]

and

\[ C^s(x) := \{(dr, d\varphi) \in \mathcal{T}_x M : -\mathcal{K}_{\text{max}} - \frac{1}{\tau_{\text{min}}} \leq \frac{d\varphi}{dr} \leq -\mathcal{K}_{\text{min}} \}. \]

Note that the angle between $C^u(x)$ and $C^s(x)$ is uniformly bounded away from zero. The cones also enjoy the following two properties.

(i) **Strict invariance.** $DT_x(C^u(x)) \subset C^u(Tx)$ and $DT_x^{-1}(C^s(x)) \subset C^s(T^{-1}x)$ if $DT$ and $DT^{-1}$ exist.

(ii) **Uniform expansion.** Let $\Lambda = 1 + 2\mathcal{K}_{\text{min}}\tau_{\text{min}}$. There exists $\hat{c} > 0$ such that

\[ \| DT_x^n(v) \| \geq \hat{c} \Lambda^n \| v \|, \forall v \in C^u(x), \quad \text{and} \quad \| DT_x^{-n}(v) \| \geq \hat{c} \Lambda^n \| v \|, \forall v \in C^s(x), \]

where $\| \cdot \|$ is the Euclidian norm. In addition, letting $T^{-1}(r, \varphi) = (r_{-1}, \varphi_{-1})$,

the expansion factor for $T^{-1}$ in the stable cone satisfies for any $x = (r, \varphi)$,

\[ C^{-1} \frac{\tau(T^{-1}x)}{\cos \varphi_{-1}} \leq \frac{\| DT_x^{-1}v \|}{\| v \|} \leq C \frac{\tau(T^{-1}x)}{\cos \varphi_{-1}}, \quad \forall v \in C^s(x), v \neq 0, \]

for some $C > 1$ independent of $x$.

Note that the expansion may not be larger than 1 at the first iteration. We can always define a norm $\| \cdot \|_*$, uniformly equivalent to $\| \cdot \|$, as an adapted norm on the tangent bundle such that (see [15, Section 5.10])

\[ \| DT_x^n(v) \|_* \geq \Lambda^n \| v \|_*, \forall v \in C^u(x), \quad \text{and} \quad \| DT_x^{-n}(v) \|_* \geq \Lambda^n \| v \|_*, \forall v \in C^s(x). \]

We say that a smooth curve $W \subset M$ is a *stable curve* if at every point $x \in W$, the tangent line $\mathcal{T}_x W$ belongs to the stable cone $C^s(x)$. We define unstable curves in the same way.

2.5.2. **Singularities.** The singularity curves of the billiard map $T$ comprise two types of curves: discontinuity curves and the boundaries of homogeneity strips.

We denote by $\mathcal{J}_0 := \{ \varphi = \pm \pi/2 \}$ the boundary of the collision space, which consists of all grazing collisions. Then the map $T$ lacks smoothness on the set $\mathcal{J}_1 := \mathcal{J}_0 \cup T^{-1} \mathcal{J}_0$. In general, denote

\[ \mathcal{J}_\pm = \bigcup_{i=0}^{n} T^{\pm i} \mathcal{J}_0. \]

For each $n = 1, 2, 3, \ldots$, the map $T^n : M \sim \mathcal{J}_n \rightarrow M \sim \mathcal{J}_n$ is a $C^2$ diffeomorphism on each connected component. The time-reversibility of $T$ implies that $\mathcal{J}_n$
and \( \mathcal{I}_n \) are symmetric about \( \varphi = 0 \) in \( M \). Moreover, the set \( \mathcal{I}_n \setminus \mathcal{I}_0 \) is a union of compact smooth stable curves for \( n \geq 1 \) and unstable curves for \( n \leq -1 \). The number of such curves is countable for billiards with infinite horizon and finite otherwise.

Each smooth curve \( S \in \mathcal{I}_n \setminus \mathcal{I}_0 \) terminates on a smooth curve in \( \mathcal{I}_n \). Furthermore, every curve \( S \in \mathcal{I}_n \setminus \mathcal{I}_0 \) is contained in one monotonically decreasing (or increasing for \( n < 0 \)) continuous curve which stretches all the way from \( \varphi = -\pi/2 \) to \( \varphi = \pi/2 \). This property is often referred to as \textit{continuation of singularity lines}.

Next we describe briefly \( \mathcal{I}_{-1} \) for the infinite-horizon case and refer to [7, 8] for more details. A point \( x \in M \) is said to be \textit{an infinite-horizon point} if the free path along its forward trajectory is infinite, or there are infinitely many consecutive grazing collisions along the trajectory of \( x \). There are only finitely many infinite-horizon points in \( M \), denoted by \( IH := \{x_1, \cdots, x_\ell\} \). By symmetry, it suffices to consider only singular curves in the upper part of \( M, \varphi \geq 0 \). In the vicinity of any \( x_i \in IH \), the set \( \mathcal{I}_{-1} \) contains a long increasing curve \( s' \) having \( x_i \) as an end-point. In addition \( \mathcal{I}_{-1} \) also contains a sequence of short increasing curves \( \{s_n\} \), connecting \( s' \) and \( \mathcal{I}_0 \), approaching \( x_i \) at the speed of order \( \Theta(1/n) \) along \( \mathcal{I}_0 \) and of order \( \Theta(1/\sqrt{n}) \) along \( s' \). More precisely, for any \( n \) large, let \( D_n \) be the cell that is bounded by \( s_n, s_{n+1}, s', \mathcal{I}_0 \). Then \( |s_n| = \Theta(1/\sqrt{n}) \), as it is almost parallel to \( s' \). There exists a constant \( C > 1 \) such that for any \( n \geq 1 \) and any point \( x \in D_n \), we have \( C^{-1}n \leq \tau(x) \leq Cn \).

In order to control distortion along stable curves, we define homogeneity strips, \( \mathbb{H}_k \), following [7]. We fix \( k_0 \in \mathbb{N} \), where \( k_0 \) is chosen so that (3.2) holds, and define for \( k \geq k_0 \),

\[
\mathbb{H}_k = \{(r, \varphi) : \pi/2 - k^{-2} < \varphi < \pi/2 - (k+1)^{-2}\}
\]

and

\[
\mathbb{H}_{-k} = \{(r, \varphi) : -\pi/2 + (k+1)^{-2} < \varphi < -\pi/2 + k^{-2}\}.
\]

We also put

\[
\mathbb{H}_0 = \{(r, \varphi) : -\pi/2 + k_0^{-2} < \varphi < \pi/2 - k_0^{-2}\}.
\]

Write

\[
S^H_{\pm k} = \{(r, \varphi) : |\varphi| = \pm \pi/2 \mp k^{-2}\} \quad \text{and} \quad \mathcal{I}_{0,H} = \mathcal{I}_0 \cup \bigcup_{k \geq k_0} S^H_{\pm k}.
\]

In general, we set \( \mathcal{I}_{\pm n}^H = \bigcup_{i=0}^n T^{\pm i} \mathcal{I}_{0,H} \) and call this the expanded singularity set for \( T^{\pm n} \). We say that a stable or unstable curve is \textit{homogeneous} if it lies entirely in one of the homogeneity strips \( \mathbb{H}_k \).

3. Preliminary Estimates and Properties of the Banach Spaces

3.1. Family of Admissible Stable Curves. Due to our definition of the stable cones \( C^2(x) \), each stable curve \( W \) can be viewed as the graph of a function \( \varphi_W(r) \) of the arc length parameter \( r \). For each stable curve \( W \), let \( I_W \) denote
the interval on which \( \varphi_W \) is defined and set \( G_W(r) = (r, \varphi_W(r)) \) to be its graph so that \( W = \{ G_W(r) : r \in I_W \} \).

We fix constants \( B > 0 \) and \( \delta_0 > 0 \), where \( \delta_0 \) is chosen small enough to satisfy the one-step expansion (3.2). We call a homogeneous stable curve admissible if \( |W| \leq \delta_0 \) and \( |\frac{d \varphi_W}{dr}| \leq B \). We define \( \mathcal{W}^s \) to be the set of admissible stable curves in \( M \). It follows directly from the uniform contraction of \( C^s(x) \) under the action of \( T^{-1} \) that if \( W \in \mathcal{W}^s \), then each (sufficiently short) component of \( T^{-1}W \) on which \( T \) is smooth is a homogeneous stable curve. It then follows from [15, Proposition 4.29] that each such smooth component is in \( \mathcal{W}^s \) if \( B \) is chosen sufficiently large.

We define an analogous family of homogeneous unstable curves \( \mathcal{W}^u \) which lie in the unstable cone \( C^u \).

Let \( W_1, W_2 \in \mathcal{W}^s \) and identify them with the graphs \( G_{W_i} \) of their functions \( \varphi_{W_i}, i = 1, 2 \). Let \( I_i := I_{W_i} \) be the \( r \)-interval on which each curve is defined and denote by \( \ell(I_1 \Delta I_2) \) the length of the symmetric difference between \( I_1 \) and \( I_2 \). Let \( \mathbb{H}_k \) be the homogeneity strip containing \( W_i \). We define the distance between \( W_1 \) and \( W_2 \) to be,

\[
d_{\mathcal{W}^s}(W_1, W_2) = \eta(k_1, k_2) + \ell(I_1 \Delta I_2) + |\varphi_{W_1} - \varphi_{W_2}|_{\varphi \in (I_1 \cap I_2)},
\]

where \( \eta(k_1, k_2) = 0 \) if \( k_1 = k_2 \) and \( \eta(k_1, k_2) = \infty \) otherwise, i.e., we only compare curves which lie in the same homogeneity strip.

Given two functions \( \psi_1, \psi_2 \in \mathcal{C}^q(W_i, C) \), we define the distance between \( \psi_1, \psi_2 \) as

\[
d_q(\psi_1, \psi_2) = |\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{\varphi \in (I_1 \cap I_2)}.
\]

We recall one final fact regarding distortion bounds for stable curves (see [15, Lemma 5.27]). Suppose that \( W \in \mathcal{W}^s \) and that \( T^i W \in \mathcal{W}^s \) for \( i = 0, 1, \ldots, n \) (i.e., each \( T^i W \)) where \( W \) is a homogeneous stable curve with uniformly bounded curvature). Then there exists \( C_d > 0 \), independent of \( n \) and \( W \), such that for any \( x, y \in W \),

\[
|\ln J_W T^n(x) - \ln J_W T^n(y)| \leq C_d d_W(x, y)^{1/3},
\]

where \( J_W T^n(x) = |\det(DT^n_\varphi)|_{\varphi \in W} \) denotes the Jacobian of \( T^n \) along \( W \) and \( d_W(\cdot, \cdot) \) is the arclength distance on \( W \).

**3.2. Growth Lemmas.** In order to prove the characterization of our Banach spaces \( \mathcal{B} \) and \( \mathcal{B}_w \) given by Lemma 2.1 as well as the estimates of Proposition 2.3, we need some understanding of the properties of \( T^{-n}W \) for \( W \in \mathcal{W}^s \). In this section we prove some growth lemmas that we shall need in Section 4.

**One Step Expansion.** Let \( W \) be a homogeneous stable curve. We partition the connected components of \( T^{-1}W \) into maximal pieces \( V_i \) such that each \( V_i \) is a homogeneous stable curve in some \( \mathbb{H}_k \). We choose \( k_0 \) large enough that the following estimate holds for both classes of billiards we consider (see [15, Lemma 5.56]):

\[
\limsup_{\delta \to 0} \sup_{|W| < \delta} \sum_i |TV_i|_+ < 1,
\]
where $|V_i|_s$ is the length of $V_i$ in the metric induced by the adapted norm $\|\cdot\|_s$. Now we choose $\delta_0$ sufficiently small that for any homogeneous stable curve $W$ with $|W| \leq \delta_0$, the sum in (3.2) is $\leq \theta_*$ for a fixed $\theta_* < 1$. In fact, by choosing $\delta_0$ sufficiently small and $k_0$ sufficiently large, one can choose $\theta_*$ arbitrarily close to $\Lambda^{-1}$ [15, eq. (5.39)]. From this point forward, we will consider $\delta_0$ and $k_0$ to be fixed by these relations. Note that this also fixes the distortion constant $C_d$ from (3.1). Next we choose $\delta_1 < \delta_0/2$ sufficiently small that

$$\delta_1 := \theta_* e^{C_d|\delta_1|^{1/3}} < \Lambda^{-1/2}. \tag{3.3}$$

To ensure that each component of $T^{-1}W$ is in $W^s$, we subdivide any of the long pieces $V_i$ whose length is $> \delta_0$. This process is then iterated so that given $W \in W^s$, we construct the components of $T^{-n}W$, which we call the $n$th generation $\mathcal{G}_n(W)$, inductively as follows. Let $\mathcal{G}_0(W) = \{W\}$ and suppose we have defined $\mathcal{G}_{n-1}(W) \subset W^s$. First, for any $W' \in \mathcal{G}_{n-1}(W)$, we partition $T^{-1}W'$ into at most countably many pieces $W'_j$ so that $T$ is smooth on each $W'_j$ and each $W'_j$ is a homogeneous stable curve. If any $W'_j$ have length greater than $\delta_0$, we subdivide those pieces into pieces of length between $\delta_0/2$ and $\delta_0$. We define $\mathcal{G}_n(W)$ to be the collection of all pieces $W'^n_j \subset T^{-n}W$ obtained in this way. Note that each $W'^n_j$ is in $W^s$ since we chose $B$ sufficiently large in the definition of $W^s$.

At each iterate of $T^{-1}$, typical short curves in $\mathcal{G}_n(W)$ grow in size, but there exist a portion of curves which are trapped in tiny Homogeneity strips and in the infinite-horizon case, stay too close to the infinite-horizon points. Our first lemma shows that the proportion of curves (in a sense made precise below) that never grow to a fixed length in $\mathcal{G}_n(W)$ decays exponentially fast.

For $W \in W^s$ and $0 \leq k \leq n$, let $\mathcal{G}_k(W) = \{W^k_j\}$ denote the $k$th-generation pieces in $T^{-k}W$. Let $B_k = \{i : |W^k_i| < \delta_1\}$ and $L_k = \{i : |W^k_i| \geq \delta_1\}$ denote the index of the short and long elements of $\mathcal{G}_k(W)$, respectively. We consider $\{\mathcal{G}_k\}_{k=0}^n$ as a tree with $W$ as its root and $\mathcal{G}_k$ as the $k$th level.

At level $n$, we group the pieces as follows. Let $W^n_{i_0} \in \mathcal{G}_n(W)$ and let $W^k_j \in L_k$ denote the most recent long “ancestor” of $W^n_{i_0}$, i.e.,

$$k = \max\{0 \leq \ell \leq n : T^{n-\ell}(W^n_{i_0}) \subset W^\ell_j \text{ and } j \in L_\ell\}.$$

If no such ancestor exists, set $k = 0$ and $W^k_j = W$. Note that if $W^n_{i_0}$ is long, then $W^k_j = W^n_{i_0}$. Let

$$\mathcal{J}_n(W^k_j) = \{i : W^k_{i,j} \in L_k \text{ is the most recent long ancestor of } W^n_{i_0}\}.$$

When $k = 0$, the set $\mathcal{J}_n(W)$ represents those curves $W^n_i \in \mathcal{G}_n(W)$ such that $T^\ell W^n_i$ belongs to a short curve in $\mathcal{G}_{n-\ell}(W)$ for each $0 \leq \ell \leq n - 1$.

**Lemma 3.1.** Let $W \in W^s$ and for $n \geq 0$, let $\mathcal{J}_n(W)$ be defined as above. There exists $C > 0$, independent of $W$, such that for any $n \geq 0$,

$$\sum_{i \in \mathcal{J}_n(W)} \frac{|T^n W^n_i|}{|W^n_i|} \leq C \theta^n_1.$$
Proof. We define a function

\[ I_n(W) = \sum_{i \in \mathcal{I}_n(W)} \frac{|T^n W_i^k|}{|W_i^k|} \]

We will show that for any admissible curve \( W \), the function \( I_n(W) \) decays exponentially as \( n \) goes to infinity. Then, since \( \| \cdot \|_\ast \) is uniformly equivalent to \( \| \cdot \| \), the lemma follows.

We prove by induction on \( n \in \mathbb{N} \) that for any \( W \in \mathcal{W}^s \), the following formula holds:

\[ I_{n+1}(W) \leq \theta_1^n \theta_\ast. \]

Note that at each iterate between 1 and \( n \), every piece \( W_i^k \), \( i \in \mathcal{I}_n(W) \), is created by genuine cuts due to singularities and homogeneity strips and not by any artificial subdivisions, since those are only made when a piece has grown to length greater than \( \delta_0 \) and \( \delta_1 \) was chosen \( < \delta_0 / 2 \). Thus we may apply the one-step expansion (3.2) to conclude,

\[ I_1(W) \leq \theta_\ast. \]

Assume that (3.5) is proved for some \( n \geq 1 \) and all \( W \in \mathcal{W}^s \). We apply it to each component \( W_i^1 \in \mathcal{B}_1(W) \) such that \( i \in \mathcal{I}_1(W) \). Then by assumption, \( I_n(W_i^1) \leq \theta_1^{n-1} \theta_\ast \), since \( W_i^1 \in \mathcal{W}^s \).

We group the components of \( W_i^{n+1} = \mathcal{B}_{n+1}(W) \) with \( i \in \mathcal{I}_{n+1}(W) \) according to elements with index in \( \mathcal{I}_1(W) \). More precisely, let \( A_k^n \) denote those indices of \( W_i^{n+1} \) such that \( T^n W_i^{n+1} \subset W_k^1 \), \( k \in \mathcal{I}_1(W) \). It follows from (3.1) that for any \( W_k^1 \), the maximum distortion of \( T \) is bounded by \( e^{C_\delta |W_k^1|} \). Thus

\[ \frac{|T^n W_i^{n+1}|}{|T^n W_i^{n+1}|} \leq e^{C_\delta |W_k^1|} \frac{|T W_k^1|}{|W_k^1|}. \]

Combining this and (3.6) with the inductive hypothesis, we get

\[
\begin{align*}
I_{n+1}(W) &= \sum_{k \in \mathcal{I}_1(W)} \sum_{i \in A_k^n} \frac{|T^n W_i^{n+1}|}{|W_i^{n+1}|} \\
&\leq \sum_{k \in \mathcal{I}_1(W)} e^{C_\delta |W_k^1|} \left( \sum_{i \in A_k^n} \frac{|T^n W_i^{n+1}|}{|W_i^{n+1}|} \right) \frac{|T W_k^1|}{|W_k^1|} \\
&= \sum_{k \in \mathcal{I}_1(W)} e^{C_\delta |W_k^1|} I_n(W_k) \cdot \frac{|T W_k^1|}{|W_k^1|} \\
&\leq \theta_1^{n-1} \theta_\ast e^{C_\delta |W_k^1|} \cdot I_1(W) \\
&\leq \theta_1^n \theta_\ast. \quad \square
\end{align*}
\]

Our next lemma allows us to iterate the control given by the one-step expansion (3.2) over pieces in \( \mathcal{B}_n(W) \).
**Lemma 3.2.** There exists $C_s > 0$, depending only on $\theta_1$, such that for any $W \in \mathcal{W}^s$ and any $n \geq 0$,

$$\sum_{W_i^n \in \mathcal{G}_n(W)} \frac{|T^n W_i^n|}{|W_i^n|} \leq C_s.$$ 

**Proof.** Fix $W \in \mathcal{W}^s$ and $n > 0$. For any $1 \leq k \leq n$, since $T^k$ is smooth on each $W_j^k \in \mathcal{G}_k(W)$, the bounded distortion (3.1) implies that if $T^{n-k} W_i^n \subset W_j^k$, then

$$\frac{|T^n W_i^n|}{|T^{n-k} W_i^n|} \leq e^{C_d \theta_1^{1/3}} \frac{|T^k W_j^k|}{|W_j^k|}. \tag{3.7}$$

Now grouping $W_i^n \in \mathcal{G}_n(W)$ by most recent long ancestor as described before the statement of Lemma 3.1 and using (3.7), we have

$$\sum_i \frac{|T^n W_i^n|}{|W_i^n|} = \sum_{k=0}^{n-1} \sum_{W_j^k \in \mathcal{G}_k(W)} \sum_{i \in \mathcal{A}_k(W)} \frac{|T^n W_i^n|}{|W_i^n|} \leq \sum_{k=1}^{n-1} \sum_{W_j^k \in \mathcal{G}_k(W)} \left( \sum_{i \in \mathcal{A}_k(W)} \frac{|T^{n-k} W_i^n|}{|W_i^n|} \right) e^{C_d \theta_1^{1/3}} \frac{|T^k W_j^k|}{|W_j^k|} + \sum_{i \in \mathcal{A}_n(W)} \frac{|T^n W_i^n|}{|W_i^n|},$$

where we have split off the sum for $k = 0$. Note that $\mathcal{A}_n(W^k)$ and $\mathcal{A}_{n-k}(W^k)$ correspond to the same set of short pieces in the $(n-k)$th generation of $W^k$, so we can apply Lemma 3.1 to each of these sums separately. Thus,

$$\sum_i \frac{|T^n W_i^n|}{|W_i^n|} \leq \sum_{k=1}^{n-1} \sum_{W_j^k \in \mathcal{G}_k(W)} C \theta_1^{n-k} e^{C_d \theta_1^{1/3}} \frac{|T^k W_j^k|}{|W_j^k|} + C \theta_1^n$$

$$\leq C \delta_1^{-1} \sum_{k=1}^{n-1} \sum_{W_j^k \in \mathcal{A}_k(W)} \theta_1^{n-k} |T^k W_j^k| + C \theta_1^n$$

$$\leq C \delta_1^{-1} |W| \sum_{k=1}^{n-1} \theta_1^{n-k} + C \theta_1^n,$$

which is uniformly bounded in $n$. \hfill \Box

The following lemma is a straightforward consequence of Lemma 3.2.

**Lemma 3.3.** Let $W \in \mathcal{W}^s$ and $0 \leq \zeta \leq 1$. Then for any $n \geq 0$,

$$\sum_{W_i^n \in \mathcal{A}_n(W)} \frac{|W_i^n|^{1-\zeta}}{|W|^{1-\zeta}} \cdot \frac{|T^n W_i^n|}{|W_i^n|} \leq C_s^{1-\zeta}.$$ 

**Proof.** Multiplying by $|W|/|W_i^n|$, we write,

$$\sum_i \frac{|W_i^n|^{1-\zeta}}{|W|^{1-\zeta}} \cdot \frac{|T^n W_i^n|}{|W_i^n|} = \sum_i \frac{|W|^{1-\zeta}}{|W_i^n|^{1-\zeta}} \cdot \frac{|T^n W_i^n|}{|W|} \leq C_s^{1-\zeta},$$

by Jensen’s inequality since $\sum_i |T^n W_i^n| |W|^{-1} = 1$. \hfill \Box
Our final result of this section concerns an extension of these results when the expansion on each piece is weakened by an exponent < 1.

**Lemma 3.4.** Let \( \varsigma > 5/6 \). There exists a constant \( C_1 = C_1(\delta_0, \varsigma) > 0 \) such that for any \( W \in \mathcal{W}^s \) and \( n \geq 0 \),

\[
\sum_{W^n_i \in \mathcal{G}_n(W)} \frac{|T^n W^n_i|^{\varsigma}}{|W^n_i|^{\varsigma}} \leq C_1^n.
\]

In the case of the finite-horizon Lorentz gas, it suffices to take \( \varsigma > 1/2 \).

**Proof.** The proof relies on the following version of the one-step expansion (3.2) for the exponent \( \varsigma > 5/6 \).

**Sublemma 3.5.** Let \( \varsigma > 5/6 \). Then there exists \( C = C(\delta_0, \varsigma) > 0 \) such that for any \( W \in \mathcal{W}^s \),

\[
\sum_i |T V_i|^{\varsigma} \leq C,
\]

where the \( V_i \)'s are the maximal homogeneous components of \( T^{-1}W \). In the case of the finite-horizon Lorentz gas, it suffices to take \( \varsigma > 1/2 \).

Before proving the sublemma, we use it to prove the following estimate by induction on \( n \):

\[
\sum_{W^n_i \in \mathcal{G}_n(W)} \frac{|T^n W^n_i|^{\varsigma}}{|W^n_i|^{\varsigma}} \leq \delta_0^{-n} C^n \varsigma^{2n}.
\]

where \( c = e^{c_{\varsigma}} \) and \( C = C(\delta_0, \varsigma) > 0 \) is from (3.8).

For \( n = 1 \): Recall that the \( W^1_i \in \mathcal{G}_1(W) \) are obtained by subdividing the maximal homogeneous components \( V_j \) of \( T^{-1}W \) of length \( > \delta_0 \). Since \( T \) is smooth with bounded distortion on each \( V_j \) and the number of \( W^1_i \) in each \( V_j \) is at most \( 1/\delta_0 \), we have by Sublemma 3.5,

\[
\sum_i |T W^1_i|^{\varsigma} \leq \sum_j e^{c_{\varsigma}} \delta_0^{-1} |T V_j|^{\varsigma} \leq \delta_0^{-1} C c.
\]

Assume at the \( n \)-th iteration, for all \( W \in \mathcal{W}^s \), we have

\[
\sum_{W^n_i \in \mathcal{G}_n(W)} \frac{|T^n W^n_i|^{\varsigma}}{|W^n_i|^{\varsigma}} \leq \delta_0^{-n} C^n \varsigma^{2n}.
\]

We group the elements \( W^{n+1}_i \in \mathcal{G}_{n+1}(W) \) according to their ancestors (long or short) \( W^1_k \in \mathcal{G}_1(W) \). More precisely, define \( A_k = \{ i : T^n W^{n+1}_i \subset W^1_k \} \). Then \( \mathcal{G}_{n+1}(W) = \bigcup_{k \geq 1} \mathcal{G}_n(W^1_k) \), where \( \mathcal{G}_n(W^1_k) := \{ W^{n+1}_i : i \in A_k \} \). Applying (3.9) to
each family $\mathcal{G}_n(W^1_{n,k})$, we obtain
\[
\sum_{W^1_{n+1} \in \mathcal{G}_{n+1}(W)} \frac{|T^{n+1}W^1_{n+1}|^\varsigma}{|W^1_{n+1}|^\varsigma} = \sum_{W^1_{n} \in \mathcal{G}_n(W)} \sum_{i \in A_k} \frac{|T^{n+1}W^1_{i}|^\varsigma}{|W^1_{i}|^\varsigma} \\
\leq \sum_{W^1_{n} \in \mathcal{G}_n(W)} \sum_{i \in A_k} e^{C_d} \frac{|T^{n}W^1_{i}|^\varsigma}{|W^1_{i}|^\varsigma} \frac{|TW^1_{k}|^\varsigma}{|W^1_{k}|^\varsigma} \\
\leq \delta_0^{-n} C_n e^{2n+1} \sum_{W^1_{n} \in \mathcal{G}_n(W)} \frac{|TW^1_{k}|^\varsigma}{|W^1_{k}|^\varsigma} \\
\leq \delta_0^{-n+1} C_n e^{2(n+1)}.
\]

Proof of Sublemma 3.5. We first prove (3.8) in the finite-horizon case and then indicate the necessary modifications in the infinite-horizon case.

Note that a stable curve of length $\leq \delta_0$ can be cut by at most $N \leq \tau_{\text{max}}/\tau_{\text{min}}$ singularity curves in $\mathcal{S}_-^\infty$ (see [15, §5.10]). For each $s \in \mathcal{S}_-^\infty$ intersecting $W$, $W$ is cut further by images of the boundaries of homogeneity strips $S^H_k$, $k \geq k_0$. For one such $s$, we relabel the components $V_i$ of $T^{-1}W$ on which $T$ is smooth by $V_k$, $k$ corresponding to the homogeneity strip $H_k$ containing $V_k$. By (2.9), there exists $c_1 > 0$ such that on $TV_k$, the expansion under $T^{-1}$ is $\geq c_1 k^2$. So for all $\varsigma > 1/2$,
\[
(3.10) \quad \sum_{k \geq k_0} \frac{|TV_k|^\varsigma}{|V_k|^\varsigma} \leq c_1^{-\varsigma} \sum_{k \geq k_0} \frac{1}{k^{2\varsigma}} \leq \frac{c_1^{-\varsigma}}{k_0^{2\varsigma-1}}.
\]

An upper bound for (3.8) in this case is given by $N$ times the bound in (3.10).

It may happen that $W$ does not intersect any $s \in \mathcal{S}_-^\infty$, but may intersect one or more preimages of the $S^H_k$. In this case, the uniform expansion in $H_0$ combined with the sum in (3.10) provides an upper bound for (3.8).

In the infinite-horizon case, in addition to the scenarios above, it may be that a stable curve $W$ intersects a countable number of singularity curves. In the notation of Section 2.5.2, assume that $W$ intersects at least two adjacent singular curves $s_n$ and $s_{n+1}$ in a neighborhood of one of the infinite-horizon points and denote the least index of the intersected $s_n \in \mathcal{S}_-^\infty$ to be $n_1$. Since $|W| < \delta_0$ and the distance between the $s_n$ along a stable curve is of order $\Theta(n^{-2})$,
\[
(3.11) \quad n_1 = \Theta(|W|^{-1/2}).
\]

According to [15, Remark 5.59], there exists $c > 0$, such that for any singular curve $s_n$ belonging to $\mathcal{S}_-$, there is a sequence $\{s_{n,k} \in TS^H_k : |k| \geq cn^{1/4}\}$ that accumulates on $s_n$ as $k$ goes to $\infty$ (or $-\infty$). We call the set bounded by $s_{n,k}, s_{n,k+1}, s'$ and $\mathcal{S}_0$, a $D_{n,k}$-cell and note that by (2.9) the expansion along stable curves under $T^{-1}$ is $\Theta(nk^2)$ in each $D_{n,k}$. Note that $D_{n,k} \subset D_n$ where $D_n$ was defined in Section 2.5.2. We relabel the components of $T^{-1}W$ as $|W_{n,k}|$ corresponding to the cell $D_{n,k}$ in which $TW_{n,k}$ lies. Then for any $\varsigma > 5/6$, we
have
\[
\sum_i \frac{|T V_i|^\zeta}{|V_i|^\zeta} \leq \sum_{n \geq n_1} \sum_{|k| \leq cn_{1/4}} \frac{|T W_{n,k}|^\zeta}{|W_{n,k}|^\zeta} \leq \sum_{n \geq n_1} \sum_{k \geq cn_{1/4}} c_1 \frac{|\zeta|}{(nk^2)\zeta} \leq C |n|^{-\frac{3}{4}\zeta + \frac{5}{8}} \leq C |\delta_0|^{-\frac{3}{4}\zeta + \frac{5}{8}} ,
\]
where we have used the relation (3.11) between $n_1$ and $|W|$.

This estimate, together with the considerations in the finite-horizon case, proves (3.8) in the infinite-horizon case.

3.3. Properties of the Banach spaces. We begin by verifying that our Banach spaces contain an interesting class of measures. We first record the following simple observation.

**Lemma 3.6.** There exists a constant $C_0 > 1$ such that for any homogeneous stable curve $W$ and any $x \in W$,
\[
C_0^{-1} \leq \frac{\cos \varphi(x)}{\cos W} \leq C_0 ,
\]
where $\varphi(x)$ is the angle at $x$ and $\cos W$ is as defined in Section 2.3. Similar bounds hold for $\cos W' / \cos W'$ whenever $W$ and $W'$ lie in the same homogeneity strip.

**Proof.** The proof is straightforward and uses the fact that
\[
\cos(\pi/2 - 1/(k + 1)^2) \leq \cos \varphi(x) \leq \cos(\pi/2 - 1/k^2)
\]
for $x \in \mathbb{H}_k$.

Our first main lemma shows that $\mathcal{B}$ contains functions with discontinuities that are transverse to the stable cone. The approximation argument rests on the fact that the contribution to the norm of a given function from homogeneity strips with high index is small.

**Lemma 3.7.** Let $\mathcal{P}$ be a (mod 0) countable partition of $M$ into open, simply connected sets such that (1) there is a constant $K > 0$ such that for each $P \in \mathcal{P}$, $\partial P$ comprises at most $K$ smooth curves, each of which is transverse to $C^\gamma(x)$, with a minimum angle uniform for all $P \in \mathcal{P}$; (2) each strip $\mathbb{H}_k$ intersects at most finitely many $P \in \mathcal{P}$.

Let $\gamma > 2\beta$. Suppose $h$ is a function on $M$ such that $\sup_{P \in \mathcal{P}} |h|_{C^\gamma(P)} < \infty$. Then $h \in \mathcal{B}$. In particular, $C^\gamma(M) \subset \mathcal{B}$ for each $\gamma > 2\beta$ and Lebesgue measure is in $\mathcal{B}$.

**Proof.** Since $\mathcal{B}$ is defined as the completion of $C^1(M)$, we must show that $h$ as above can be approximated by functions in $C^1(M)$ in the $\|\cdot\|_{\mathcal{B}}$ norm.

For $P \in \mathcal{P}$ we define $P_k$ to be a single simply connected component of $P \cap \mathbb{H}_k$. The labeling may not be unique, but there are only finitely many elements of $\mathcal{P}$ labelled $P_k$ for each $k \geq k_0$ by assumption (2) on $\mathcal{P}$.

Let $h$ be as in the statement of the lemma. Since $\|h\|_{\mathcal{B}} = \sup_{P \in \mathcal{P}} \|h|_{\mathbb{H}_k}\|_{\mathcal{B}}$ by definition of $W^\phi$, we may fix $k$ and approximate $h$ one $\mathbb{H}_k$ at a time. We fix $P_k$ and for simplicity first consider $h \equiv 0$ off of $P_k$. 


Choose $\eta > 0$ such that $\tilde{P}_k := B_{\eta/k^3}(P_k)$, the $\eta/k^3$ neighborhood of $P_k$, satisfies $\tilde{P}_k \subset H_{k-1} \cup H_k \cup H_{k+1}$ for $H_{k_0}$, we use $k = k_0$. Choose a smooth foliation of stable curves on $\tilde{P}_k$ and extend $h$ to the smaller neighborhood $B_{\eta/(2k^3)}(P_k)$ by extending $h$ as a constant function along each stable curve in the foliation. Denote this extended function by $\tilde{h}_k$ and set it equal to 0 elsewhere.

Let $\rho_{\eta}(x, y)$ be a nonnegative $C^\infty$ bump function such (1) $\int_{\tilde{P}_k} \rho_{\eta}(x, y) \, d m(y) = 1$ for each $x \in \tilde{P}_k$, and (2) $\rho_{\eta}(x, y) = 0$ whenever $d(x, y) > \eta/(2k^3)$. Define

$$f_{\eta}(x) = \int_{\tilde{P}_k} \tilde{h}_k(y) \rho_{\eta}(x, y) \, d m(y), \text{ for } x \in M.$$ 

Note that $f_{\eta} \in C^\infty(M)$ and that $f_{\eta}(x) \equiv 0$ for $x \in \tilde{P}_k$. We may also arrange it so that $|f_{\eta}|_{C^1(P_k)} \leq |h|_{C^1(P_k)}$, while $|f_{\eta}|_{C^\gamma(M)} \leq C|\eta|_{C^\gamma(P_k)}k^{3\gamma}/\eta^\gamma$ for some $C > 0$ independent of $k$ and $\eta$.

Now let $W \in \mathcal{W}^3$, $W \subset H_k$, and take $\psi \in C^q(W)$ with $|\psi|_{W,a,q} \leq 1$. Note that $|\psi|_{\infty} \leq |W|^{-a}(\cos W)^{-1}$. Thus,

$$\int_W (h - f_{\eta}) \psi \, d m_W = \int_{W \cap P_k} (h - f_{\eta}) \psi \, d m_W + \int_{W \setminus P_k} (h - f_{\eta}) \psi \, d m_W$$

$$\leq |h - f_{\eta}|_{C^q(W \cap P_k)} |W|^{-a}(\cos W)^{-1} + |f_{\eta}|_{\infty}(\text{supp } f_{\eta}) \cap (W \setminus P_k) ||W||^{-a}(\cos W)^{-1}.$$ 

(3.12)

For the first term above, we estimate the difference in functions for $x \in W \cap P_k$ by,

$$|h(x) - f_{\eta}(x)| \leq \int_{P_k} |h(x) - \tilde{h}_k(y)| \rho_{\eta}(x, y) \, d m(y)$$

and note that we only need consider $y$ such that $d(x, y) \leq \eta/(2k^3)$ by definition of $\rho_{\eta}$, i.e., $y$ such that $\tilde{h}_k(y) = h(z)$ for some $z \in P_k$ by definition of $\tilde{h}_k$. Also, since $\partial P_k$ is transverse to $C^\gamma(x)$ and $h$ was extended along stable curves, we have $d(y, z) \leq C\eta/(2k^3)$. Thus $d(x, z) \leq C\eta/k^3$ and so

$$|h(x) - f_{\eta}(x)| \leq C|h|_{C^\gamma(P_k)} \eta \gamma k^{-3\gamma}.$$ 

Consider the expression $|W|^{-a}(\cos W)^{-1}$. Since $W$ is a homogeneous curve, it lies either in $H_{k_0}$ or in a homogeneity strip indexed by $k \geq k_0$. In the former case, $\cos W \geq 1/k_0^3$ so that the above expression is bounded. In the latter case, $\cos W \geq 1/k^2$ and $|W| \leq C k^{-3}$ since the stable cone is uniformly transverse to the boundaries of the homogeneity strips. Thus

$$|W|^{-a}(\cos W)^{-1} \leq C k^{3(\alpha - 1)}k^2 < C k^{-1/2},$$ 

(3.13)

since $\alpha < 1/6$. Putting these estimates together, we obtain for the first term of (3.12),

$$|h - f_{\eta}|_{C^q(W \cap P_k)} |W|^{-a}(\cos W)^{-1} \leq C|h|_{C^\gamma(P_k)} \eta \gamma k^{-1/2}.$$ 

For the second term of (3.12), we consider two cases.

Case 1: $|W| < \eta/k^3$. Then $(\text{supp } f_{\eta}) \cap (W \setminus P_k) \subset |W|$ so that using (3.13),

$$|f_{\eta}|_{\infty}(\text{supp } f_{\eta}) \cap (W \setminus P_k) ||W||^{-a}(\cos W)^{-1} \leq C|h|_{\infty} \eta^{-a} k^{-1/2}.$$
Case 2: $|W| > \eta/k^3$. Then since $|(\text{supp} f_\eta) \cap (W \sim P_k)| < \eta/(2k^3)$, we have
\[
|f_\eta|_{\infty}(\text{supp} f_\eta) \cap (W \sim P_k)||W|^{-\alpha} (\cos W)^{-1} \leq C|h|_{\infty}\eta(2k^3)^{-1}(\eta/k^3)^{-\alpha}k^2 \\
\leq C|h|_{\infty}^{-1}\alpha k^{-1/2}.
\]
Putting together these estimates and taking the suprema over $W \subset \mathbb{H}_k$ and $\psi \in \mathcal{C}^q(W)$, we have by (3.12),
\[
\|(h - f_\eta^{p_k})|_{\mathbb{H}_k}\|_s \leq C|h|_{\infty}(\eta^\gamma + \eta^{-1-\alpha})k^{-1/2}.
\]
Note that if we were not concerned with approximation, (3.12) and (3.13) would together with our estimate on $C_q W$ and $\eta$, we have by (3.13)
\[
\|(h - f_\eta^{p_k})|_{\mathbb{H}_k}\|_s \leq C|h|_{\infty}k^{-1/2}
\]
for all bounded functions $h$.

Since $f_\eta$ is supported on $\mathbb{H}_{k-1} \cup \mathbb{H}_k \cup \mathbb{H}_{k+1}$, we must estimate the norm of $h - f_\eta$ on $\mathbb{H}_{k+1}$ as well. Recalling that $h \equiv 0$ on $M \sim P_k$ and $f_\eta \equiv 0$ on $M \sim \tilde{P}_k$, for $W \subset \mathbb{H}_{k+1}$ and $\psi|_{W,a,q} \equiv 1$ we estimate,
\[
\int_W (h - f_\eta) \psi \, dm_W \leq |f_\eta|_{\infty}|W \cap \tilde{P}_k||W|^{-\alpha}(\cos W)^{-1} \leq C|h|_{\infty}\eta^{-\alpha}k^{-1/2},
\]
again using (3.13) and cases 1 and 2 above since $|W \cap \tilde{P}_k| \leq C\eta/k^3$. Putting this together with our estimate on $\mathbb{H}_k$, we have $\|(h - f_\eta)|_{\mathbb{H}_k}\|_s \leq C|h|_{\infty}(\eta^\gamma + \eta^{-1-\alpha})k^{-1/2}$.

To estimate $\|(h - f_\eta)|_{\mathbb{H}_k}\|_s$, we fix $0 < \epsilon \leq \epsilon_0$, where $\epsilon_0$ is from (2.3), and let $W_1, W_2 \subset \mathbb{H}_k$ be two admissible stable curves such that $d_{\mathbb{H}}(W_1, W_2) \leq \epsilon$. In the notation of Section 3.1, we identify $W_i$ with the graph $G_{W_i}$ of its defining function $\psi_{W_i}(r)$, $r \in I_i$. Let $\psi_1, \psi_2$ be two test functions satisfying $|\psi_i|_{W_i,0,p} \leq 1$, $i = 1, 2$, and $|\psi_1 \circ G_{W_1} - \psi_2 \circ G_{W_2}|_{\mathcal{C}(\bar{W}_1 \cap I_2)} \leq \epsilon$. Without loss of generality, assume $\gamma = 2\beta + \delta \leq 1/2$, for some $\delta > 0$. This is always possible since $\beta < 1/6$ by definition of the norms.

First assume that $\epsilon \geq \eta^2k^{-1/3}$. Then by the estimate on the stable norm, we have
\[
\epsilon^{-\beta}\int_{W_1} (h - f_\eta) \psi_1 \, dm_W - \int_{W_2} (h - f_\eta) \psi_2 \, dm_W \leq C\epsilon^{-\beta} |h|_{\mathcal{C}^1(P)} \eta^\gamma k^{-1/2} \\
\leq C\eta^\delta |h|_{\mathcal{C}^1(P)}.
\]

It remains to estimate the case $\epsilon < \eta^2k^{-1/2}$. For this estimate, we split up the terms involving $h$ and $f_\eta$.

\[
(3.15) \quad \int_{W_1} (h - f_\eta) \psi_1 \, dm_W - \int_{W_2} (h - f_\eta) \psi_2 \, dm_W \\
= \int_{W_1} h\psi_1 \, dm_W - \int_{W_2} h\psi_2 \, dm_W + \int_{W_2} f_\eta \psi_2 \, dm_W - \int_{W_1} f_\eta \psi_1 \, dm_W.
\]

We first estimate the difference involving $h$.

We match $W_1$ and $W_2$ using a foliation of vertical line segments of length at most $\epsilon$ wherever possible. This partitions $W_1$ in the following way: curves $U_1^j \subset W_1$ for which the vertical segment connecting $U_1^j$ to $W_2$ lies entirely in $P_k$; curves $V_1^j \subset W_1$ which either are not matched to $W_2$ (near the end-points of
We call $U^j_k$ the matched pieces and $V^j_k$ the unmatched pieces and note that by assumption on $\mathcal{P}$, there can be no more than $K$ matched pieces and $K+2$ unmatched pieces.

We split up the integrals on $W_1$ and $W_2$ on matched and unmatched pieces,

\begin{equation}
W_1 \end{equation}

We estimate the integrals on the unmatched pieces first. Since $h \equiv 0$ off of $P_k$ and $\partial P_k$ and the vertical lines are both uniformly transverse to the stable cone (see property (1) of $\mathcal{P}$ in the statement of the lemma), we have $|\text{supp}(h) \cap V^j_k| \leq C\varepsilon$ for each $V^j_k$. Then using (3.14), we estimate

\begin{equation}
W_2 \end{equation}

where in the last inequality, $|\psi_k|_{\mathcal{C}(W_k)} \leq (\cos W_k)^{-1}$ and we have used Lemma 3.6 to bound $\cos V^j_k / \cos W_k$.

Next we estimate the difference on matched pieces in (3.16). To do this, we change variables to the $r$ intervals $I_i$ common to $U^j_k$ and $U^j_m$.

\begin{equation}
W_3 \end{equation}

where $JG_{U^j_k}$ denotes the Jacobian of $G_{U^j_k}$. Note that

\begin{equation}
W_4 \end{equation}

We split the difference on matched pieces into the sum of three terms. The first term is,

\begin{equation}
W_5 \end{equation}

where $H^r(h)$ denotes the Hölder constant of $h$ with exponent $r$ on $P_k$. Now $d(G_{U^j_k}(r),G_{U^j_m}(r)) = |\phi_{U^j_k}(r) - \phi_{U^j_m}(r)| \leq \varepsilon$ by definition of $d^r(\cdot,\cdot)$. Thus,

\begin{equation}
W_6 \end{equation}

The second term of the difference is,

\begin{equation}
W_7 \end{equation}
by assumption on $\psi_1$ and $\psi_2$. Finally, the last difference we must estimate is,

\[(3.21) \quad E := |h \circ G_{U_2} \psi_2 \circ G_{U_2} |_{\infty} |JG_{U_2} |_{\infty} - |JG_{U_1} |_{\infty} \]

\[\leq |h|_{\infty} |\psi_2|_{\infty} |\psi'_{U_1} - \psi'_{U_2}|_{\infty} \leq |h|_{\infty} |\epsilon| \cos W_2,
\]

again by definition of $d_{\psi,\cdot}(\cdot, \cdot)$, where $\psi'_{U_k} = \frac{d\psi_{U_k}}{dx}$.

Putting together the estimates for $A$, $B$ and $E$, as well as (3.17), into (3.16), we have

\[(3.22) \quad e^{-\beta} \left| \int_{W_1} h\psi_1 \, dm_W - \int_{W_2} h\psi_2 \, dm_W \right| 
\leq C|h|_{\ell^\gamma(P_k)} |W_1| \left( \frac{e^{\gamma-\beta}}{\cos W_1} + \frac{e^{1-\beta}}{\cos W_2} \right) + C|h|_{\infty} e^{a-\beta} k^{-1/2}.
\]

Note that the estimate (3.22) holds without the assumption $\epsilon < \eta^2 k^{-\frac{1}{2\beta}}$ which is what makes (3.24) below possible.

A similar estimate holds for $f_{\eta}$, although now we use the assumption $\epsilon < \eta^2 k^{-\frac{1}{2\beta}}$. Indeed the estimate is simpler since $f_{\eta}$ is Hölder-continuous on all of $M$ with $H^\gamma(f_{\eta}) \leq C|h|_{\ell^\gamma(P_k)} k^{\frac{2\gamma}{\eta}}$. Thus we may partition $W_1$ and $W_2$ into one matched piece and at most two unmatched pieces near their end-points. The unmatched pieces have length at most $C\epsilon$ so that an estimate similar to (3.17) holds for $f_{\eta}$. Then since $f_{\eta}$ is Hölder-continuous everywhere, estimates $A$, $B$ and $E$ hold on the single matched piece and so,

\[(3.23) \quad e^{-\beta} \left| \int_{W_1} f_{\eta}\psi_1 \, dm_W - \int_{W_2} f_{\eta}\psi_2 \, dm_W \right| 
\leq C|W_1| \left( \frac{H^\gamma(h) e^{\gamma-\beta} k^{2\gamma}}{\eta^\gamma \cos W_1} + \frac{|h|_{\infty} e^{1-\beta}}{\cos W_2} \right) + C|h|_{\infty} e^{a-\beta}.
\]

Since $|W_1|/\cos W_1$ is bounded by $C_k$ by (3.13) and $\cos W_1 / \cos W_2 \leq C_0$ by Lemma 3.6 because $W_1$ and $W_2$ lie in the same homogeneity strip, it is clear that the only term that can cause a problem is the first one in (3.23). We estimate,

\[\frac{|W_1| \cdot \frac{e^{\gamma-\beta} k^{2\gamma}}{\eta^\gamma}}{\cos W_1} \leq C_k \frac{1}{\eta^\gamma \cdot k^{(\gamma-\beta)/(2\beta)}} \leq C \eta^{\delta} k^{(6\gamma - 6 - \beta)/2\beta}.
\]

Note that the exponent of $k$ is negative since $6\beta \gamma < \gamma < \gamma + \beta$ for any $\gamma > 0$ and $\beta < 1/6$.

We have shown that $\| (h - f_{\eta}) |_{\ell^\gamma} \|_u \leq C|h|_{\ell^\gamma(P_k)} \eta^{\delta'}$, where $\delta' = \min(\delta,2(\alpha-\beta))$. Since $h \equiv 0$ outside $P_k$, we have $\| (h - f_{\eta}) |_{H^{k\varepsilon_1}} \|_u = \| f_{\eta} |_{H^{k\varepsilon_1}} \|_u$ and this expression is similarly bounded by (3.23) since the bound on $H^\gamma(f_{\eta})$ used there holds on all of $M$.

This together with the estimate on the strong stable norm implies that $\| h - f_{\eta} \|_{\ell^\beta} \leq C|h|_{\ell^\gamma(P_k)} \eta^{\delta'}$. Note that if we are not concerned with approximation,
then (3.13), (3.14) and (3.22) together imply that

\[(3.24) \quad \|h\|_{H_k} \leq C \sup_{P \in \mathcal{P}} |h|_{C^1(T(P))} k^{-1/2}.\]

In making this approximation argument, we have assumed that \( h \equiv 0 \) outside \( P_k \). More general \( h \) can be expressed as \( h = \sum_k \sum P_k h1_{P_k} \) where \( h1_{P_k} \equiv 0 \) outside of \( P_k \) and so can be approximated by a \( \mathcal{C}^1 \) function \( f^p_k \) as above. Due to (3.24), given \( \epsilon > 0 \), we first choose \( \varepsilon' \) so that \( \|h|_{H_k}\|_{\mathcal{B}} < \epsilon \) for all \( k > K' \). By property (2) of \( \mathcal{P} \) in the statement of the lemma, there exists a constant \( N_{\varepsilon} \) such that each strip \( H_k \) for \( k_0 \leq k \leq K' \) intersects at most \( N_{\varepsilon} \) elements \( P \in \mathcal{P} \). We thus form the finite sum \( \sum_{k_0} \sum_{k \leq K'} \sum P_k f^p_k \) and approximate \( h \) by \( 0 \) on \( \bigcup_{k > K'} \mathcal{B}_k \). Note that there are at most \( N_{\varepsilon} \) elements \( P_k \) for each \( k \). Thus,

\[ \| (h - \sum_{k_0} \sum_{k \leq K'} \sum P_k f^p_k) \|_{\mathcal{B}} \leq \epsilon + \sup_{k_0 \leq k \leq K'} \| \sum P_k (h1_{P_k} - f^p_k) \|_{\mathcal{B}} \leq C N_{\varepsilon} \eta \eta' \sup_{P \in \mathcal{P}} |h|_{C^1(T(P))}, \]

and finally we choose \( \eta \) sufficiently small so that \( \eta \eta' N_{\varepsilon} < \epsilon \). \( \square \)

Our next lemma shows that \( \mathcal{L} \) is well-defined as an operator from \( \mathcal{B} \) to \( \mathcal{B} \). Its proof uses the fact that \( \|\mathcal{L}h\|_{\mathcal{B}} < \infty \) from Section 4. This is the only point in this section where we use results from Section 4.

**Lemma 3.8.** If \( h \in \mathcal{C}^1(M) \), then \( \mathcal{L} h \in \mathcal{B} \).

**Proof.** Let \( h \in \mathcal{C}^1(M) \). As in the proof of Lemma 3.7, we must approximate \( \mathcal{L} h \) by \( \mathcal{C}^1 \) functions in the norm \( \| \cdot \|_{\mathcal{B}} \). Note that \( \mathcal{L} h \) has a countable number of smooth discontinuity curves given by \( T(\mathcal{A}_0, h) \) (we include the images of boundaries of the homogeneity strips). These curves define a countable partition \( \mathcal{P} \) of \( M \) into open simply connected sets which does not satisfy the assumption (2) of Lemma 3.7 since each \( H_k \) can intersect countably many \( P \in \mathcal{P} \).

In addition, the \( \mathcal{C}^1 \) norm of \( \mathcal{L} h \) blows up near the curves \( T\mathcal{A}_0 \).

For \( j \geq k_0 \) let \( P^j \) denote an element of \( \mathcal{P} \) such that \( T^{-1} P^j \subseteq H_j \). Again, the labeling is not unique, but for each \( j \), the number of elements in \( \mathcal{P} \) which are assigned the label \( j \) is finite (even in the infinite-horizon case). Let \( P^j = \bigcup_{j \geq j} P^j \). We claim that \( \|\mathcal{L} h|_{P^j}\|_{\mathcal{B}} \) is arbitrarily small for \( j \) sufficiently large. On the finite set of \( P^j \) with \( j \leq j \), the \( \mathcal{C}^1 \) norm of \( \mathcal{L} h \) is finite and the modified partition \( \mathcal{P}^* = \{P^j\}_{j \leq j} \cup \{P^j\} \) satisfies the requirements of Lemma 3.7. So we may approximate \( \mathcal{L} h \) as in Lemma 3.7 on \( M \sim P^j \) and approximate \( \mathcal{L} h \) by \( 0 \) on \( P^j \). Thus the lemma follows once we establish our claim.

Indeed, the claim is trivial using the estimates of Section 4. For example, we must estimate \( \|\mathcal{L} h|_{P^j}\|_{\mathcal{B}} = \|1_{P^j} \mathcal{L} h\|_{\mathcal{B}} \). Taking \( W \in \mathcal{W}^s \) and \( \psi \in \mathcal{C}^q(W) \) with \( |\psi|_{W^s,a,q} \leq 1 \), we write

\[ \int_W 1_{P^j} \mathcal{L} h \psi \, dm_W = \int_{T^{-1}(W \cap P^j)} h |DT|^{-1} J_{T^{-1}W} T \psi \circ T \, dm_W, \]

and the homogeneous stable components of \( T^{-1}(W \cap P^j) \) correspond precisely to the tail of the series considered in (4.2) and following and so can be made arbitrarily small by choosing \( J \) large (note that we do not need contraction here.
so that we may use the simpler estimate similar to Section 4.1 applied to the
strong stable norm rather than the estimate of Section 4.2).

Similarly, in estimating \( \| \mathcal{L} h \|_{L^p} \), one can see that the contribution from \( P^J \) corresponds to the tail of the series from the estimates of Section 4.3, and so this too can be made arbitrarily small by choosing \( J \) large.

The next lemma allows us to establish a connection between our Banach spaces and the space of distributions introduced in Section 2.2. Recall that 

\[
H^p_W(\psi) = \sup_{W^p \cap \mathbb{R}^d} H^p_W(\psi).
\]

**Lemma 3.9.** For each \( h \in C^1(M) \), \( n \geq 0 \), and \( \psi \in C^0(T^{-n}W^s) \) we have

\[
|h(\psi)| = \left| \int_M h \psi \, dm \right| \leq C |h|_\infty (|\psi|_\infty + |H^p_W(\psi)|).
\]

**Proof.** On each \( M_\ell = \partial \Gamma_\ell \times [-\pi/2, \pi/2] \), we partition the set \( \mathcal{H}_0 \cap M_\ell \) into finitely many boxes \( B_j \) whose boundary curves are straight line segments in \( W^s \) and \( W^u \) as well as the horizontal lines \( \pm \pi/2 \) \( 1/k_0^2 \). We construct the boxes so that each \( B_j \) has diameter \( \leq \delta_0 \) and is foliated by curves \( W \in W^s \). On each \( B_j \), we choose a smooth foliation \( \{ W_\xi \} \) \( \subset W^s \) of parallel straight line segments, each of whose elements completely crosses \( B_j \) in the approximate stable direction (this is always possible if we originally choose the stable and unstable boundaries of \( B_j \) to be parallel).

We decompose Lebesgue measure on \( B_j \) into \( dm = \lambda(d\xi) dm_{W_\xi} \), where \( m_{W_\xi} \) is the conditional measure of \( m \) on \( W_\xi \) and \( \lambda \) is the transverse measure on \( E_j \).

We normalize the measures so that \( m_{W_\xi}(W_\xi) = |W_\xi| \) and note that the conditional measure \( m_{W_\xi} \) is the arclength measure on \( W_\xi \) since the foliation is comprised of straight line segments. Note also that \( \lambda(E_j) \leq C \delta_0 \) due to the transversality of curves in \( W^s \) and \( W^u \).

Next we foliate each homogeneity strip \( \mathcal{H}_k \cap M_\ell \), \( k \geq k_0 \), using a smooth family of parallel line segments \( \{ W_\xi \} \) \( \subset W^s \) whose elements all have end-points lying in the two boundary curves of \( \mathcal{H}_k \). We again decompose \( m \) on \( \mathcal{H}_k \) into \( dm = \lambda(d\xi) dm_{W_\xi} \), \( \xi \in E_k \), and \( m_{W_\xi}(W_\xi) = |W_\xi| \) is normalized as above. By construction, \( \lambda(E_k) = \Theta(1) \).

Now given \( h \in C^1(M) \) and \( \psi \in C^0(T^{-n}W^s) \), note that since \( M = T^{-n}M \) (mod 0), \( \int_M h \psi \, dm = \int_M \mathcal{L}^n h \psi \circ T^{-n} \, dm \). We estimate the second integral one \( \ell \) at a time,

\[
\int_{M_\ell} \mathcal{L}^n h \psi \circ T^{-n} \, dm = \sum_j \int_{B_j} \mathcal{L}^n h \psi \circ T^{-n} \, dm + \sum_{|k| \geq k_0} \int_{\mathcal{H}_k \cap M_\ell} \mathcal{L}^n h \psi \circ T^{-n} \, dm = \sum_j \int_{E_j} \int_{W_\xi} \mathcal{L}^n h \psi \circ T^{-n} \, dm_{W_{\xi}} \, d\lambda(\xi) + \sum_{|k| \geq k_0} \int_{E_k} \int_{W_\xi} \mathcal{L}^n h \psi \circ T^{-n} \, dm_{W_{\xi}} \, d\lambda(\xi).
\]

We change variables and estimate the integrals on one \( W_\xi \) at a time. Letting \( W^n_\xi \), denote the components of \( \mathcal{G}_n(W_\xi) \) defined in Section 3.2, we define
Let \( J_{W^n} T^n \) to be the stable Jacobian of \( T^n \) along the curve \( W^n_\xi \), and write
\[
\int_{W^n_\xi} \mathcal{L}^n h \circ T^{-1} d m_W = \sum_i \int_{W^n_\xi} h(x) \cdot |DT^n|^{-1} J_{W^n_\xi} T^n d m_W \\
\leq \sum_i |h|_{\infty} \cdot \cos(W^n_\xi) \cdot |\psi_\xi(W^n_\xi)| \cdot |DT^n|^{-1} J_{W^n_\xi} T^n |\psi_\xi(W^n_\xi)|.
\]

From the distortion bounds (A.1) we have
\[
||DT^n||^{-1} J_{W^n_\xi} T^n |\psi_\xi(W^n_\xi)| \leq C^2 ||DT^n||^{-1} J_{W^n_\xi} T^n |\psi_\xi(W^n_\xi)|.
\]

Since by [15, (2.29)], the Jacobian \(|DT^n||^{-1}(x) = \cos(T^n(x)) / \cos x \) for \( x \in W^n_\xi \), we have by Lemma 3.6,
\[
\cos(W^n_\xi) ||DT^n||^{-1} |\psi_\xi(W^n_\xi)| \leq C_2 \cos W_\xi.
\]

Also by (3.1), \(|J_{W^n_\xi} T^n| \cdot \psi_\xi(W^n_\xi) \leq e^{-\delta_0 \sqrt{n} ||W^n_\xi|| ||W^n_\xi||}. \) Putting these estimates together yields
\[
\int_{W^n_\xi} \mathcal{L}^n h \circ T^{-1} d m_W \leq C|h|_{\infty} (|\psi|_{\infty} + H^n p(\psi)) \cdot \sum_i \frac{|T^n W^n_\xi|}{W^n_\xi},
\]
where the sum is \( \leq C_\delta \) by Lemma 2. Thus
\[
\left| \int_{M_\ell} \mathcal{L}^n h \circ T^{-1} d m \right| \leq C|h|_{\infty} (|\psi|_{\infty} + H^n p(\psi)) \cdot \left( \sum_j \int_{E_j} \cos W_\xi d \lambda(\xi) + \sum_{|k| \geq k_0} \int_{E_k} \cos W_\xi d \lambda(\xi) \right) \\
\leq C|h|_{\infty} (|\psi|_{\infty} + H^n p(\psi)) \left( \sum_j \lambda(E_j) + \sum_{|k| \geq k_0} k^{-2} \lambda(E_k) \right),
\]
where in the last line we have used the fact that \( \cos W \leq Ck^{-2} \) for \( W \subset H_k \). Both sums are finite since there are only finitely many \( E_j \) and \( \lambda(E_k) \) is of order 1 for each \( k \). Since there are only finitely many \( M_\ell \), we may sum over \( \ell \) and the lemma is proved. \( \square \)

We conclude this section by proving the following important fact.

**Lemma 3.10.** The unit ball of \( \mathcal{B} \) is compactly embedded in \( \mathcal{B}_w \).

**Proof.** First note that on a fixed \( W \in \mathcal{W}^s \), \( |\cdot|_{W,a,q} \) is equivalent to \( |\cdot|_{\psi(W)} \) and \( |\cdot|_{W,a,q} \) is equivalent to \( |\cdot|_{\psi(W)} \) so that \( p > q \) implies that the unit ball of \( |\cdot|_{W,a,p} \) is compactly embedded in \( |\cdot|_{W,a,q} \). Since \( \|\cdot\|_s \) is the dual of \( |\cdot|_{W,a,q} \) and \( |\cdot|_w \) is the dual of \( |\cdot|_{W,a,p} \) on each stable curve \( W \in \mathcal{W}^s \), the unit ball of \( \|\cdot\|_s \) is compactly embedded in \( |\cdot|_w \) on \( W \). It remains to compare the weak norm on different stable curves.

We argue one component \( M_\ell = \Gamma_\ell \times [-\pi/2, \pi/2] \) at a time. Let \( 0 < \epsilon \leq \epsilon_0 \) be fixed. Let \( k_\epsilon \in \mathbb{N} \) be the first integer \( k \) such that \( 1/k^2 < \epsilon \). We split \( M_\ell \) into two parts, \( A = [-\pi/2 + 1/k_\epsilon^2, \pi/2 - 1/k_\epsilon^2] \) and \( B = M_\ell \setminus A \). Since curves in \( \mathcal{W}^s \) are graphs of functions \( \varphi_W \) whose slopes are greater than \( \lambda_{\min} > 0 \) and have
uniformly bounded second derivative, there exists \( C = C(Q) > 0 \) such that any admissible curve \( W \subset B \) must have length no longer than \( C \varepsilon \).

Let \( h \in \mathcal{C}^1(M) \) with \( \|h\|_{\mathcal{B}} \leq 1 \). First we estimate the weak norm of \( h \) on curves \( W \) in \( B \). If \( W \subset H_k \) for \( |k| \geq k_\varepsilon \), and \( |\psi_j|_{W,0,p} \leq 1 \), then

\[
\int_W h\psi \, dm_W \leq \|h\|_{\mathcal{B}} |W|^\alpha \cos W |\psi\|_{\mathcal{E}_\gamma(W)} \leq C \|h\|_{\mathcal{B}} \varepsilon^\alpha.
\]

Now for \( W \subset A \), note that there exists a constant \( K_\varepsilon > 1 \) such that \( 1/\cos W \leq K_\varepsilon \). On a fixed interval \( I \), the set of functions \( \{q_W\}_{W \in \mathcal{W}} \), defined on \( I \) and lying in one homogeneity strip is compact in the \( \mathcal{C}^1 \)-norm. Since \( A \) contains only finitely many homogeneity strips, we may choose finitely many stable curves \( W_i \in \mathcal{W}^\varepsilon \) such that \( \{W_i\}_{i=1}^{N_\varepsilon} \) forms an \( \varepsilon \)-covering of \( \mathcal{W}^\varepsilon \) in the distance \( d_{\mathcal{W}^\varepsilon} \).

Let \( |\Gamma_\ell| \) denote the arclength of \( \Gamma_\ell \) and define \( \mathbb{S}_\ell^1 \) to be the circle of length \( |\Gamma_\ell| \). Since any ball of finite radius in the \( \mathcal{C}^p \)-norm is compactly embedded in \( \mathcal{C}^q \), we may choose finitely many functions \( \overline{\psi}_j \in \mathcal{C}^p(\mathbb{S}_\ell^1) \) such that \( \{\overline{\psi}_j\}_{j=1}^{N_\varepsilon} \) forms an \( \varepsilon \)-covering in the \( \mathcal{C}^q(\mathbb{S}_\ell^1) \)-norm of the ball of radius \( C_g K_\varepsilon \) in \( \mathcal{C}^p(\mathbb{S}_\ell^1) \), where \( C_g \) is from (3.18).

Now let \( W = G_W(I_W) \in \mathcal{W}^\varepsilon \mid A \), and \( \psi \in \mathcal{C}^p(W) \) with \( |\psi|_{W,0,p} \leq 1 \). We view \( I_W \) as a subset of \( \mathbb{S}_\ell^1 \). Let \( \overline{\psi} = \psi \circ G_W \) be the push down of \( \psi \) to \( I_W \). Note that \( \|\overline{\psi}\|_{\mathcal{P}(I_W)} = C_\varepsilon / \cos W \leq C_g K_\varepsilon \).

Choose \( W_i = G_W(I_W_i) \) such that \( d_{\mathcal{W}^\varepsilon}(W, W_i) \leq \varepsilon \) and choose \( \overline{\psi}_j \in \mathcal{C}^p(\mathbb{S}_\ell^1) \) such that \( |\overline{\psi} - \overline{\psi}_j|_{\mathcal{C}^q(I_W)} \leq \varepsilon \). Define \( \overline{\psi}_j = \overline{\psi}_j \circ G_{W_i}^{-1} \) to be the lift of \( \overline{\psi}_j \) to \( W_i \). Define \( |\psi_j|_{W_i,0,p} \leq \cos W_i (2C_g / \cos W) \leq 2C_g C_\varepsilon \) by Lemma 3.6 since \( W_i \) and \( W \) lie in the same homogeneity strip. Then normalizing \( \psi \) and \( \psi_j \) by \( 2C_0 C_g \), we estimate

\[
\left| \int_W h\psi \, dm_W - \int_{W_i} h\psi_j \, dm_W \right| \leq \varepsilon^\beta \|h\|_{\mathcal{B}} 2C_0 C_g.
\]

We have proved that for each \( 0 < \varepsilon \leq \varepsilon_0 \), there exist finitely many bounded linear functionals \( \ell_{i,j} \), \( \ell_{i,j}(h) = \int_W h\psi_j \, dm_W \), such that

\[
|h|_{W} \leq \max_{i \leq N, j \leq L_\varepsilon} \ell_{i,j}(h) + \varepsilon^\beta C \|h\|_{\mathcal{B}} + \varepsilon^{\alpha} C \|h\|_{\mathcal{B}} \leq \max_{i \leq N, j \leq L_\varepsilon} \ell_{i,j}(h) + \varepsilon^\beta C b^{-1} \|h\|_{\mathcal{B}},
\]

which implies the required compactness.

\[ \square \]

4. Lasota–Yorke Estimates

It suffices to prove Proposition 2.3 for \( h \in \mathcal{C}^1(M) \) since then by density of \( \mathcal{C}^1(M) \) in \( \mathcal{B} \), \( \mathcal{L} \) is continuous on \( \mathcal{B} \). To see this, assume Proposition 2.3 has been proved for \( h \in \mathcal{C}^1(M) \) and identify \( h \in \mathcal{B} \) with a Cauchy sequence \( \{g_n\}_{n \geq 0} \subset \mathcal{C}^1(M) \). Since \( \mathcal{L} \) is bounded when applied to functions in \( \mathcal{C}^1(M) \), by the assumption that Proposition 2.3 holds for \( \mathcal{C}^1 \) functions, it follows that \( \{\mathcal{L} g_n\} \) is a Cauchy sequence in \( \mathcal{B} \). By Lemma 3.9, we identify its limit with \( \mathcal{L} h \) and so \( \|\mathcal{L} h\|_{\mathcal{B}} = \lim_n \|\mathcal{L} g_n\|_{\mathcal{B}} \leq \lim_n C \|g_n\|_{\mathcal{B}} = C \|h\|_{\mathcal{B}} \). Thus \( \mathcal{L} \) is bounded and therefore continuous on \( \mathcal{B} \). A similar argument holds for \( \mathcal{B}_W \).

We use the distortion bounds of Appendix A throughout this section.
4.1. **Estimating the Weak Norm.** Let \( h \in \mathcal{C}^1(M) \), \( W \in \mathcal{W}^s \) and \( \psi \in \mathcal{C}^p(W) \) such that \( |\psi|_{W,0,p} \leq 1 \). For \( n \geq 0 \), we write,

\[
\int_W \mathcal{L}^n h \psi \, dm_W = \sum_{W_i \in \mathcal{G}_n(W)} \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|} \psi \circ T^n \, dm_W,
\]

where \( J_{W_i^n} T^n \) denotes the Jacobian of \( T^n \) along \( W_i^n \).

Using the definition of the weak norm on each \( W_i^n \), we estimate (4.1) by

\[
\int_W \mathcal{L}^n h \psi \, dm_W \leq \sum_{W_i^n \in \mathcal{G}_n} |h|_w ||DT^n||^{-1} J_{W_i^n} T^n |\psi \circ T^n|_{\mathcal{C}^p(W_i^n)} \cos W_i^n.
\]

The distortion bounds given by equation (A.1) imply that

\[
||DT^n||^{-1} J_{W_i^n} T^n |\psi \circ T^n|_{\mathcal{C}^p(W_i^n)} \leq C_d^2 ||DT^n||^{-1} J_{W_i^n} T^n |\psi \circ T^n|_{\mathcal{C}^p(W_i^n)}.
\]

For \( x, y \in W_i^n \), we record for future use,

\[
\frac{|\psi(T^n x) - \psi(T^n y)|}{d_W(T^n(x), T^n(y))^p} \frac{d_W(T^n(x), T^n(y))^p}{d_W(x, y)^p} \leq C|\psi|_{\mathcal{C}^p(W)} J_{W_i^n} T^n |\psi \circ T^n|_{\mathcal{C}^p(W_i^n)} \leq C\Lambda^p |\psi|_{\mathcal{C}^p(W)}
\]

by (2.8) so that \( |\psi \circ T^n|_{\mathcal{C}^p(W^n_i)} \leq C|\psi|_{\mathcal{C}^p(W)} \leq C/\cos W \). Using these estimates in equation (4.2), we obtain

\[
\int_W \mathcal{L}^n h \psi \, dm_W \leq C|h|_w \sum_{W_i^n \in \mathcal{G}_n} \frac{\cos W_i^n}{\cos W} ||DT^n||^{-1} J_{W_i^n} T^n |\psi \circ T^n|_{\mathcal{C}^p(W_i^n)}.
\]

Since \( |DT^n(x)| = \cos \varphi(x)/\cos \varphi(T^n x) \) for \( x \in W_i^n \), by Lemma 3.6, we have

\[
||DT^n||^{-1} C^n(W_i^n) \cos W_i^n \cos W \leq C_d^2.
\]

Note also that by the bounded distortion estimate (3.1),

\[
|J_{W_i^n} T^n|_{\mathcal{C}^0(W_i^n)} \leq e^{C_\delta |\delta|^3 T^n W_i^n ||W_i^n||^{-1}}.
\]

Gathering these estimates together, we obtain

\[
\int_W \mathcal{L}^n h \psi \, dm_W \leq C|h|_w \sum_{W_i^n \in \mathcal{G}_n} \frac{|T^n W_i^n|}{|W_i^n|} \leq C C_d |h|_w,
\]

where in the last inequality we have used Lemma 3.2. Taking the supremum over all \( W \in \mathcal{W}^s \) and \( \psi \in \mathcal{C}^p(W) \) with \( |\psi|_{W,0,p} \leq 1 \) yields (2.4).

4.2. **Estimating the Strong Stable Norm.** Let \( W \in \mathcal{W}^s \) and let \( W_i^n \) denote the elements of \( \mathcal{G}_n(W) \) as defined in Section 3.2. For \( \psi \in \mathcal{C}^q(W) \), \( |\psi|_{W,a,q} \leq 1 \), define \( \mathcal{W}_i = |W_i^n|^{-1} \int_{W_i^n} \psi \circ T^n \, dm_W \). Using equation (4.1), we write

\[
\int_W \mathcal{L}^n h \psi \, dm_W = \sum_i \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|} (\psi \circ T^n - \mathcal{W}_i) \, dm_W + \mathcal{W}_i \int_{W_i^n} h \frac{J_{W_i^n} T^n}{|DT^n|} \, dm_W.
\]
To estimate the first term of (4.5), we first estimate \(|\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^q(W^n_i)}\). If \(H^q_W(\psi)\) denotes the Hölder constant of \(\psi\) along \(W\), then equation (4.3) implies

\[
(4.6) \quad \frac{|\psi(T^n x) - \psi(T^n y)|}{d_W(x, y)^q} \leq C\Lambda^{-q} H^q_W(\psi),
\]

for any \(x, y \in W^n_i\). Since \(\overline{\psi}_i\) is constant on \(W^n_i\), we have \(H^q_W(\psi \circ T^n - \overline{\psi}_i) \leq C\Lambda^{-q} H^q_W(\psi)\). To estimate the \(\mathcal{C}^q\) norm, note that \(\overline{\psi}_i = \psi \circ T^n(y_i)\) for some \(y_i \in W^n_i\). Thus for each \(x \in W^n_i\),

\[
|\psi \circ T^n(x) - \overline{\psi}_i| = |\psi \circ T^n(x) - \psi \circ T^n(y_i)| \leq H^q_W(\psi \circ T^n)|W^n_i|^q \leq CH^q_W(\psi)\Lambda^{-q}.
\]

This estimate together with (4.6) and the fact that \(|\varphi|_{W, a, q} \leq 1\), implies

\[
(4.7) \quad |\psi \circ T^n - \overline{\psi}_i|_{\mathcal{C}^q(W^n_i)} \leq C\Lambda^{-q} |\psi|_{\mathcal{C}^q(W)} \leq C\Lambda^{-q} |W|^{-\alpha} (\cos W)^{-1}.
\]

We apply (4.7), the distortion estimate (A.1) and the definition of the strong stable norm to the first term on the right-hand side of (4.5),

\[
(4.8) \quad \sum_i \int_{W^n_i} h \frac{J^n_i}{|DT^n|} (\psi \circ T^n - \overline{\psi}_i) \, dm_W
\]

\[
\leq C \sum_i \|h\|_s \frac{|W^n_i|^\alpha \cos W^n_i}{|W|^\alpha \cos W} \left| \frac{J^n_i}{|DT^n|} \right|_{C^q(W^n_i)} \Lambda^{-q} n
\]

\[
\leq C e^{\|h\|_s} C_0 \Lambda^{-q} n \sum_i \frac{|W^n_i|^\alpha |T^n_i W_i^n|}{|W|^\alpha |W_i^n|} \leq C \Lambda^{-q} \|h\|_s,
\]

where in the second line we have used (4.4) and Lemma 3.3 with \(\zeta = \alpha\).

For the second term of (4.5), we use the fact that \(|\overline{\psi}_i| \leq |W|^{-\alpha} (\cos W)^{-1}\) since \(|\psi|_{W, a, q} \leq 1\). Recall the notation introduced before the statement of Lemma 3.1. Grouping the pieces \(W^n_i \in \mathcal{G}_n(W)\) according to most recent long ancestors, we have

\[
\sum_i \frac{1}{|W|^\alpha \cos W} \int_{W^n_i} h \frac{J^n_i}{|DT^n|} \, dm_W
\]

\[
= \sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in \mathcal{G}_n(W^n_j)} \frac{1}{|W|^\alpha \cos W} \int_{W^n_i} h \frac{J^n_i}{|DT^n|} \, dm_W
\]

\[
+ \sum_{i \in \mathcal{G}_n(W)} \frac{1}{|W|^\alpha \cos W} \int_{W^n_i} h \frac{J^n_i}{|DT^n|} \, dm_W,
\]

where we have split up the terms involving \(k = 0\) and \(k \geq 1\). We estimate the terms with \(k \geq 1\) by the weak norm and the terms with \(k = 0\) by the strong
stable norm,

$$\sum_1^n \frac{1}{|W|^a \cos W} \int_{W_k} h J_{W_i}^n T^n \frac{J_{W_i}^n T^n}{|D^n|} \, dm_W$$

$$\leq C \sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in \mathcal{F}_i(W_k^j)} \frac{\cos W_i^n}{|W|^a \cos W} |h|_w J_{W_i}^n \frac{J_{W_i}^n T^n}{|D^n|} \|h\|_s |D^n|^{-1} J_{W_i}^n T^n |\phi(W_i^n)|$$

$$+ C \sum_{i \in \mathcal{F}_i(W)} |W_i^n|^a \cos W_i^n \|h\|_s |D^n|^{-1} J_{W_i}^n T^n |\phi(W_i^n)|.$$  

As usual, by (4.4), the ratio of cosines times $|D^n|^{-1}$ is uniformly bounded. In the first sum above corresponding to $k \geq 1$, we write

$$|J_{W_i}^n T^n |\phi(W_i^n)| \leq |J_{W_i}^n T^{n-k} |\phi(W_i^n)| J_{W_j} T^k |\phi(W_j^k)|.$$  

Thus using Lemma 3.1 from time $k$ to time $n$,

$$\sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in \mathcal{F}_i(W_j^k)} |W_i^n|^a J_{W_i}^n T^n |\phi(W_i^n)| \leq C \sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in \mathcal{F}_i(W_j^k)} \frac{|T^{n-k} W_i^n|}{|W_i^n|}$$

$$\leq C \delta_1^n \sum_{k=1}^n \sum_{j \in L_k} \sum_{i \in \mathcal{F}_i(W_j^k)} \frac{|T^{k} W_j^k|}{|W_j^k|} \frac{|W_j^k|^a}{|W_i^n|} \theta_i^{n-k},$$

since $|W_j^k| \geq \delta_1$. The last two sums are bounded independently of $n$ and $W$ by Lemma 3.3 with $\zeta = \alpha$.

Finally, for the sum corresponding to $k = 0$, we write

$$\sum_{i \in \mathcal{F}_i(W)} \frac{|W_i^n|^a}{|W|^a} J_{W_i}^n T^n |\phi(W_i^n)| \leq C \left( \sum_{i \in \mathcal{F}_i(W)} \frac{|T^n W_i^n|}{|W_i^n|} \right)^{1-\alpha} \leq C \theta_1^{n(1-\alpha)},$$

using Lemma 3.1 and Jensen’s inequality as in the proof of Lemma 3.3.

Gathering these estimates together, we have

$$(4.9) \quad \sum_\ell \frac{1}{|W|^a \cos W} \left| \int_{W_i^n} h |D^n|^{-1} J_{W_i}^n T^n \, dm_W \right| \leq C \delta_1^{-\alpha} |h|_w + C \|h\|_s \theta_1^{n(1-\alpha)}.$$  

Putting together (4.8) and (4.9) proves (2.5):

$$\|\mathcal{L}^n h\|_s \leq C \left( \Lambda^{-\alpha} + \theta_1^{(1-\alpha)} \right) \|h\|_s + C \delta_1^{-\alpha} |h|_w.$$  

4.3. Estimating the Strong Unstable Norm. Fix $\epsilon \leq \epsilon_0$ and consider two curves $W^1, W^2 \in \mathcal{W}$ with $d_{\mathcal{W}}(W^1, W^2) \leq \epsilon$. For $n \geq 1$, we describe how to partition $T^{-n} W^\ell$ into “matched” pieces $U_i^\ell$ and “unmatched” pieces $V_i^\ell$, $\ell = 1, 2$.

Let $\omega$ be a connected component of $W^1 \sim \mathcal{F}_{\delta_1}$. To each point $x \in T^{-n} \omega$, we associate a vertical line segment $\gamma_x$ of length at most $C \Lambda^{-n} \epsilon$ such that its image
$T^n \gamma_x$, if not cut by a singularity or the boundary of a homogeneity strip, will have length $C \varepsilon$. By [15, §4.4], all the tangent vectors to $T^i \gamma_x$ lie in the unstable cone $C^u(T^i x)$ for each $i \geq 1$ so that they remain uniformly transverse to the stable cone and enjoy the minimum expansion given by (2.8).

Doing this for each connected component of $W^1 \sim \mathcal{F}_n^H$, we subdivide $W^1 \sim \mathcal{F}_n^H$ into a countable collection of subintervals of points for which $T^n \gamma_x$ intersects $W^2 \sim \mathcal{F}_n^H$ and subintervals for which this is not the case. This in turn induces a corresponding partition on $W^2 \sim \mathcal{F}_n^H$.

We denote by $V^\ell$ the pieces in $T^{-n} W^\ell$ which are not matched up by this process and note that the images $T^n V^\ell$ occur either at the end-points of $W^\ell$ or because the vertical segment $\gamma_x$ has been cut by a singularity. In both cases, the length of the curves $T^n V^\ell$ can be at most $C \varepsilon$ due to the uniform transversality of $\mathcal{F}_n^H$ with the stable cone and of $C^u(x)$ with $C^u(x)$.

In the remaining pieces the foliation $\{T^n \gamma_x\}_x \in T^{-n} W^1$ provides a one-to-one correspondence between points in $W^1$ and $W^2$. We further subdivide these pieces in such a way that the lengths of their images under $T^{-i}$ are less than $\delta_0$ for each $0 \leq i \leq n$ and the pieces are pairwise matched by the foliation $\gamma_x$. We call these matched pieces $U^\ell_j$. Possibly changing the constant $\delta_0/2$ to $\delta_0/C$ for some uniform constant $C > 1$ (depending only on the distortion constant and the angle between stable and unstable cones) in the definition of $\mathcal{G}_n(W^\ell)$, we may arrange it so that $U^\ell_j \subset W^\ell_j$ for some $W^\ell_j \in \mathcal{G}_n(W^\ell)$ and $V^\ell \subset W^\ell_j$ for some $W^\ell_j \in \mathcal{G}_n(W^\ell)$ for all $j, k \geq 1$ and $\ell = 1, 2$. There is at most one $U^\ell_j$ and two $V^\ell_j$ per $W^\ell_j \in \mathcal{G}_n(W^\ell)$.

In this way we write $W^\ell = (\bigcup_j T^n U^\ell_j) \cup (\bigcup_j T^n V^\ell_j)$. Note that the images $T^n V^\ell_j$ of the unmatched pieces must be short while the images of the matched pieces $U^\ell_j$ may be long or short.

Recalling the notation of Section 3.1, we have arranged a pairing of the pieces $U^\ell_j$ with the following property:

$$\text{If } U^\ell_j = G_{U^\ell_j}(I_j) = \{(r, \varphi_{U^\ell_j}(r)) : r \in I_j\},$$

then $U^\ell_j = G_{U^\ell_j}(I_j) = \{(r, \varphi_{U^\ell_j}(r)) : r \in I_j\},$

so that the point $x = (r, \varphi_{U^\ell_j}(r))$ is associated with the point $\tilde{x} = (r, \varphi_{U^\ell_j}(r))$ by the vertical segment $\gamma_x \subset \{(r, s) \in [-\pi/2, \pi/2]\}$, for each $r \in I_j$.

**Remark 4.1.** The fact that we have matched stable curves using vertical line segments is not essential to our argument: we could have matched them using any smooth foliation of curves in the unstable cones. However, a remarkable feature of the present approach is that we do not match stable curves along real unstable manifolds, as is commonly done in coupling arguments, and thus we avoid the technical difficulties associated with the corresponding holonomy map.
Given $\psi_\ell$ on $W^\ell$ with $|\psi_\ell|_{W^{\ell,0,p}} \leq 1$ and $d_q(\psi_1, \psi_2) \leq \varepsilon$, with the above construction we must estimate

\[(4.11)\]
\[
\int_{W^1} \mathcal{L}^n h \psi_1 \, dm_W - \int_{W^2} \mathcal{L}^n h \psi_2 \, dm_W \leq \sum_{\ell,i} \left| \int_{V_i^\ell} h|DT^n|^{-1} J_{V_i^\ell} T^n \psi_\ell \circ T^n \, dm_W \right| + \sum_{j} \left| \int_{U_j^1} h|DT^n|^{-1} J_{U_j^1} T^n \psi_1 \circ T^n \, dm_W - \int_{U_j^2} h|DT^n|^{-1} J_{U_j^2} T^n \psi_2 \circ T^n \, dm_W \right|.
\]

We do the estimate over the unmatched pieces $V_i^\ell$ first using the strong stable norm. Note that by (4.3), $|\psi_\ell|_{T^n,\mathcal{Q}(T^{-n}V_i^\ell)} \leq C|\psi_\ell|_{T^n(W_i^\ell)} \leq C(\cos W^\ell)^{-1}$. We estimate as in Section 4.2, using the fact that $|T^n V_i^\ell| \leq C\varepsilon$,

\[(4.12)\]
\[
\sum_{\ell,i} \left| \int_{V_i^\ell} h|DT^n|^{-1} J_{V_i^\ell} T^n \psi_\ell \circ T^n \, dm_W \right| \leq C\sum_{\ell,i} \|h\|_s |V_i^\ell|^\alpha \|DT^n|^{-1} J_{V_i^\ell} T^n|_{|T^n V_i^\ell|} \cos V_i^\ell \cos W^\ell
\]
\[
\leq C\|h\|_s \sum_{\ell,i} |V_i^\ell|^\alpha |T^n V_i^\ell| \leq C\varepsilon \sum_{\ell,i} |T^n V_i^\ell|^{1-\alpha} \leq C\varepsilon \|h\|_s C_1^n,
\]

where we have applied Lemma 3.4 with $\zeta = 1 - \alpha > 5/6$ since there are at most two $V_i^\ell$ corresponding to each element $W_i^{\ell,n} \in \mathcal{Q}_i^n(W)$ as defined in Section 3.2 and by bounded distortion, $|T^n V_i^\ell| \leq C|T^n W_i^{\ell,n}|$.

Next, we must estimate

\[
\sum_{j} \left| \int_{U_j^1} h|DT^n|^{-1} J_{U_j^1} T^n \psi_1 \circ T^n \, dm_W - \int_{U_j^2} h|DT^n|^{-1} J_{U_j^2} T^n \psi_2 \circ T^n \, dm_W \right|.
\]

Recalling the notation defined by (4.10), we fix $j$ and estimate the difference. Define

$$\phi_j = (|DT^n|^{-1} J_{U_j^1} T^n \psi_1 \circ T^n) \circ G_{U_j^1} \circ G_{U_j^2}^{-1}.$$ 

The function $\phi_j$ is well-defined on $U_j^2$ and we can estimate

\[(4.13)\]
\[
\int_{U_j^1} h|DT^n|^{-1} J_{U_j^1} T^n \psi_1 \circ T^n - \int_{U_j^2} h|DT^n|^{-1} J_{U_j^2} T^n \psi_2 \circ T^n \leq \left| \int_{U_j^2} h|DT^n|^{-1} J_{U_j^2} T^n \psi_1 \circ T^n - \int_{U_j^2} h\phi_j \right| + \left| \int_{U_j^2} h(\phi_j - |DT^n|^{-1} J_{U_j^2} T^n \psi_2 \circ T^n) \right|.
\]
We estimate the first term in equation (4.13) using the strong unstable norm. The distortion bounds given by (A.1) and the estimates of (4.3) and (4.4) imply that

\[
|DT^n|^{-1} J_{U_j} T^n \cdot \psi_1 \circ T^n |_{\psi(U_j)} \leq \cos(U_j^1) |DT^n|^{-1} J_{U_j} T^n \cdot \psi_1 \circ T^n |_{\psi(U_j)} \\
\leq C \frac{\cos(U_j^1)}{\cos W^1} |DT^n|^{-1} J_{U_j} T^n |_{\psi(U_j)} \\
\leq C |J_{U_j} T^n |_{\psi(U_j)}.
\]

(4.14)

Similarly, since \(|G_{U_j} \circ G_{U_j}^{-1}|_{\psi(U_j)} \leq C_g\), where \(C_g\) is from (3.18),

\[
\cos(U_j^2) |\phi_j|_{\psi(U_j)} \leq C \frac{\cos(U_j^2)}{\cos W^2} |DT^n|^{-1} J_{U_j} T^n |_{\psi(U_j)} \leq C |J_{U_j} T^n |_{\psi(U_j)}
\]

where \(\frac{\cos(U_j^1)}{\cos W^1} \leq C_0 \frac{\cos(U_j^1)}{\cos W^1}\) by Lemma 3.6 since the corresponding curves lie in the same homogeneity strips. By the definition of \(\phi_j\) and \(d_q(\cdot, \cdot)\),

\[
d_q(|DT^n|^{-1} J_{U_j} T^n \psi_1 \circ T^n, \phi_j) = \left| \left| |DT^n|^{-1} J_{U_j} T^n \psi_1 \circ T^n \right| - \left| \left| G_{U_j} - \phi_j \circ G_{U_j^2} \right|_{\psi(U_j)} \right| = 0.
\]

To complete the estimate on the first term of (4.13), we need the following lemma.

**Lemma 4.2.** There exists \(C > 0\), independent of \(W_1\) and \(W_2\), such that for each \(j\),

\[
d_{\psi(U_j)}(U_j, U_j) \leq CA^{-n}ne = : \epsilon_1.
\]

We postpone the proof of the lemma until Section 4.3.1 and use it to complete the estimate of the first term of (4.13).

In view of (4.14), we renormalize the test functions by \(R_j = C |J_{U_j} T^n |_{\psi(U_j)}\).

Then we apply the definition of the strong unstable norm with \(\epsilon_1\) in place of \(\epsilon\). Thus,

\[
\sum_j \left| \int_{U_j} h |DT^n|^{-1} J_{U_j} T^n \psi_1 \circ T^n - \int_{U_j} h \phi_j \right| \\
\leq C \epsilon_1 \left\| h \right\| u \sum_j |J_{U_j} T^n |_{\psi(U_j)} \leq C \left\| h \right\| u A^{-n} \epsilon_1 \epsilon \sum_j \frac{|T^n U_j|}{|U_j|},
\]

where the sum is \(\leq C_s\) by Lemma 3.2 since there is at most one matched piece \(U_j^1\) corresponding to each component of \(T^{-n} W_1\), \(W_i^{1,n} \in \mathcal{G}_n(W_1)\).

It remains to estimate the second term in (4.13) using the strong stable norm. We need the following lemma.
LEMMA 4.3. There exists $C > 0$ such that for each $j \geq 1$ one has

\[
\| (|DT^n|^{-1} J^n_1 T^n) \circ G_{U_j} - (|DT^n|^{-1} J^n_2 T^n) \circ G_{U_j} \|_{C^0(\bar{U}_j)} \leq C \| DT^n \|^{-1} J^n_1 T^n \|_{C^0(\bar{U}_j)} C^{1/3 - q}.
\]

Proof. Throughout the proof, for ease of notation we write $J^n_\epsilon$ for $|DT^n|^{-1} J^n_1 T^n$.

For any $r \in I_j$, $x = G_{U_j}(r)$ and $\bar{x} = G_{U_j}(\epsilon)$ lie on a common vertical segment $\gamma_x$. Thus $T^n(x)$ and $T^n(\bar{x})$ also lie on the element $T^n \gamma_x \in \mathcal{W}$ which intersects $W^1$ and $W^2$ and has length at most $Ce$. By (4.3) and (4.4),

\[
|J^n_1(x) - J^n_2(\bar{x})| \leq C |J^n_1|_{C^0(U_j)} |d(T^n x, T^n \bar{x})|^{1/3} + \theta(T^n x, T^n \bar{x}),
\]

where $\theta(T^n x, T^n \bar{x})$ is the angle between the tangent line to $W^1$ at $T^n x$ and the tangent line to $W^2$ at $T^n \bar{x}$. Let $y \in W^2$ be the unique point in $W^2$ which lies on the same vertical segment as $T^n x$. Since by assumption $d_{\mathcal{W}}(W^1, W^2) \leq \epsilon$, we have $\theta(T^n x, y) \leq \epsilon$. Due to the uniform transversality of curves in $\mathcal{W}$ and $\mathcal{W}^s$ and the fact that $W^1$ and $W^2$ are graphs of $C^2$ functions with uniformly bounded $C^2$ norms, we have $\theta(y, T^n \bar{x}) \leq Ce$ and so $\theta(T^n x, T^n \bar{x}) \leq Ce$. Similarly, $d_W(T^n x, T^n \bar{x}) \leq Ce$ so that

\[
|J^n_1(x) - J^n_2(\bar{x})| \leq Ce^{1/3} |J^n_2|_{C^0(U_j)}.
\]

Using this estimate and the fact that $|G_{U_j}|_{C^1(I_j)} \leq C$, we write for $r, s \in I_j$,

\[
|J^n_1 \circ G_{U_j}(r) - J^n_2 \circ G_{U_j}(s)| = 2Ce^{1/3} |J^n_2|_{C^0(U_j)} |r - s|^{1/3 - q}.
\]

Also, using (A.1) since $G_{U_j}(r)$ and $G_{U_j}(s)$ lie on the same stable curve,

\[
|J^n_1 \circ G_{U_j}(r) - J^n_2 \circ G_{U_j}(s)| = 2C |J^n_2|_{C^0(U_j)} |r - s|^{1/3 - q}.
\]

Putting (4.17) and (4.18) together implies that the Hölder constant of $J^n_1 \circ G_{U_j} - J^n_2 \circ G_{U_j}$ is bounded by

\[
H^q(j_1 \circ G_{U_j} - j_2 \circ G_{U_j}) \leq C |J^n_2|_{C^0(U_j)} \sup_{r, s \in I_j} |r - s|^{-q}.
\]

This expression is maximized when $\epsilon^{1/3} |r - s|^{-q} = |r - s|^{1/3 - q}$, i.e., when $\epsilon = |r - s|$. Thus the Hölder constant satisfies, $H^q(j_1 \circ G_{U_j} - j_2 \circ G_{U_j}) \leq C |J^n_2|_{C^0(U_j)} \epsilon^{1/3 - q}$, which, together with (4.16), concludes the proof of the lemma. \qed

Using the strong stable norm, we estimate the second term in (4.13) by

\[
\int_{U_j^2} h(\phi_j - |DT^n|^{-1} J^n_2 T^n \psi_2 \circ T^n)
\]

\[
\leq \| h \|_{L^1} |U_j^2|^{1/3} \cos(U_j^2) |\phi_j - |DT^n|^{-1} J^n_2 T^n \psi_2 \circ T^n|_{C^0(U_j)}.
\]
In order to estimate the $\mathcal{E}^q$-norm of the function in (4.19), we split it up into two differences. Since $|G_{U_j}|_{\mathcal{E}^1}, |G_{U_j}^{-1}|_{\mathcal{E}^1} \leq C\varepsilon$, we obtain

\begin{align}
|\phi_j - (|DT^n|^{-1} J_{U_j})_n \cdot \psi_2 \circ T^n|_{\mathcal{E}^p} & \leq C \left| (|DT^n|^{-1} J_{U_j} T^n) \cdot \psi_1 \circ T^n \circ G_{U_j} - (|DT^n|^{-1} J_{U_j} T^n) \cdot \psi_2 \circ T^n \circ G_{U_j} \right|_{\mathcal{E}^q} \\
& \leq C \left| (|DT^n|^{-1} J_{U_j} T^n) \circ G_{U_j} \left[ \psi_1 \circ T^n \circ G_{U_j} - \psi_2 \circ T^n \circ G_{U_j} \right] \right|_{\mathcal{E}^q(I_j)} \\
& \quad + C \left| (|DT^n|^{-1} J_{U_j} T^n) \circ G_{U_j} - (|DT^n|^{-1} J_{U_j} T^n) \circ G_{U_j} \right|_{\mathcal{E}^q(I_j)} \\
& \leq C |DT^n|^{-1} J_{U_j} T^n|_{\mathcal{E}^q(U_j)} \left| \psi_1 \circ T^n \circ G_{U_j} - \psi_2 \circ T^n \circ G_{U_j} \right|_{\mathcal{E}^q(I_j)} \\
& \quad + C (\cos W^2)^{-1} \left| (|DT^n|^{-1} J_{U_j} T^n) \circ G_{U_j} - (|DT^n|^{-1} J_{U_j} T^n) \circ G_{U_j} \right|_{\mathcal{E}^q(I_j)}.
\end{align}

Note that the second term can be bounded using Lemma 4.3. To bound the first term, we prove the following lemma.

**Lemma 4.4.** $|\psi_1 \circ T^n \circ G_{U_j} - \psi_2 \circ T^n \circ G_{U_j}|_{\mathcal{E}^q(U_j)} \leq C (\cos W^2)^{-1} \varepsilon^{p-q}$.

We postpone the proof of the lemma to Section 4.3.1 and show how this completes the estimate on the strong unstable norm. Note that

$|DT^n|^{-1} J_{U_j} T^n|_{\mathcal{E}^q(U_j)} \leq C |DT^n|^{-1} J_{U_j} T^n|_{\mathcal{E}^q(U_j)}$

by the distortion bounds (A.3) and (A.4). Then using Lemmas 4.3 and 4.4 together with (4.20) yields by (4.19)

\begin{align}
\sum_j \left| \int_{U_j} h(\phi_j - |DT^n|^{-1} J_{U_j} T^n \psi_2 \circ T^n) \, dm_W \right| \\
& \leq C \|h\| \sum_j |U_j^2|^{\alpha \cos U_j^2 / \cos W^2} |DT^n|^{-1} J_{U_j} T^n|_{\mathcal{E}^q(U_j)} \varepsilon^{p-q} \\
& \leq C \|h\| \varepsilon^{p-q} \sum_j \frac{|T^n U_j^2|}{|U_j^2|},
\end{align}

where again the sum is finite as in (4.15). This completes the estimate on the second term in (4.13). Now we use this bound, together with (4.12) and (4.15) to estimate (4.11)

\begin{align}
\left| \int_{W_1} \mathcal{L}^n h \psi_1 \, dm_W - \int_{W_2} \mathcal{L}^n h \psi_2 \, dm_W \right| \\
& \leq CC_4^n \|h\| \varepsilon^{p-q} + C \|h\| \Lambda^{-n\beta} n^\beta \varepsilon + C \|h\| \varepsilon^{p-q}.
\end{align}

Since $p - q \geq \beta$ and $\alpha \geq \beta$, we divide through by $\varepsilon^{\beta}$ and take the appropriate suprema to complete the proof of (2.6).
4.3.1. Proof of Lemmas 4.2 and 4.4.

Proof of Lemma 4.2. Note that by construction $U^1_j$ and $U^2_j$ lie in the same homogeneity strip. Also, they are both defined on the same interval $I_j$ so the length of the symmetric difference of their $r$-intervals is 0. Recalling the definition of $d_{W^s}(U^1_j, U^2_j)$, we see that it remains only to estimate $|\varphi_{U^1_j} - \varphi_{U^2_j}|_{\varphi^1(I_j)}$ for their defining functions $\varphi_{U^1_j}$.

The fact that $|\varphi_{U^1_j} - \varphi_{U^2_j}|_{\varphi^1(I_j)} \leq C \Lambda^{-n} \varepsilon$ follows from the fact that $U^1_j$ and $U^2_j$ are connected by a foliation of vertical segments $|\gamma_x|$ and $T^i\gamma_x$ lies in the enlarged unstable cone $\hat{C}^u(x) = \{(dr, d\varphi) \in \mathcal{S}_xM : \mathcal{K}_\min \leq \frac{d\varphi}{dr} \leq \infty\}$,

\begin{align*}
\text{for } 0 \leq i \leq n \text{. Any vector in } \hat{C}^u(x) \text{ undergoes the uniform expansion}^3 \text{ given by (2.8) under iteration by } T \text{ (see [15, §4.4]) and } |T^n\gamma_x| \leq C\varepsilon \text{ by assumption on } W^1 \text{ and } W^2, \text{ we have } |\gamma_x| \leq C \Lambda^{-n} \varepsilon.
\end{align*}

Finally, we must estimate $|\varphi_{U^1_j}' - \varphi_{U^2_j}'|$, where $\varphi_{U^1_j}'$ denotes the derivative of $\varphi_{U^1_j}$ with respect to $r$. For $x \in U^1_j$, let $\phi(x)$ denote the angle that $G^r_{U^1_j}$ makes with the positive $r$-axis at $x$. For $x \in U^1_j$ and $\bar{x} = \gamma_x \cap U^2_j$, let $\theta(x, \bar{x})$ denote the angle between the tangent vectors to $U^1_j$ and $U^2_j$ at the points $x$ and $\bar{x}$, respectively. We have

\begin{align*}
|\varphi_{U^1_j}'(x) - \varphi_{U^2_j}'(\bar{x})| &= |\tan \phi(x) - \tan \phi(\bar{x})| \\
&\leq \left[ \sup_{z \in U^1_j} \sec^2 \phi(z) \right] |\phi(x) - \phi(\bar{x})| \\
&= \left[ \sup_{z \in U^1_j} \sec^2 \phi(z) \right] \theta(x, \bar{x}).
\end{align*}

Since the slopes of vectors in $C^r(x)$ are uniformly bounded away from 0 and $-\infty$, we have $\sec^2 \phi(z)$ uniformly bounded above for any $z \in U^1_j$. It follows from (A.5) that

$\theta(x, \bar{x}) \leq C \Lambda^{-n} (n d(T^n x, T^n \bar{x}) + \theta(T^n x, T^n \bar{x})).$

Since $T^n x \in W^1$, $T^n \bar{x} \in W^2$, it follows from the assumption $d_{W^s}(W^1, W^2) \leq \varepsilon$ that $d(T^n x, T^n \bar{x}) + \theta(T^n x, T^n \bar{x}) \leq C\varepsilon$, which proves the lemma.

Proof of Lemma 4.4. Let $\varphi_{W^r}$ be the function whose graph is $W^r$, defined for $r \in I_{W^r}$, and set $f^r_j := G^{-1}_{W^1} \circ T^n \circ G_{U^1_j}$, $\ell = 1, 2$. Note that since $|G_{W^1}^{-1}|_{\varphi^1}, |G_{U^1_j}|_{\varphi^1} \leq C_g$, and due to the uniform contraction along stable curves, we have $|f^r_j|_{\varphi^1(I_j)} \leq C$, where $C$ is independent of $W^r$ and $j$. We may assume that $f^r_j(I_j) \subset I_{W^1} \cap I_{W^2}$ since if not, by the transversality of $C^u(x)$ and $C^s(x)$, we must be in a neighborhood of one of the end-points of $W^r$ of length at most $C\varepsilon$; such short pieces of
may be estimated as in (4.12) using the strong stable norm. Thus

\begin{equation}
|\psi_1 \circ T^n \circ G_{f_j} - \psi_2 \circ T^n \circ G_{f_j}|_{\mathcal{E}^q(I_j)} \\
\leq |\psi_1 \circ G_{W^1} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^1|_{\mathcal{E}^q(I_j)} + |\psi_2 \circ G_{W^2} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^2|_{\mathcal{E}^q(I_j)}.
\end{equation}

Using the above observation about \( f_j^1 \), we estimate the first term of (4.22) by

\begin{equation}
|\psi_1 \circ G_{W^1} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^1|_{\mathcal{E}^q(I_j)} \leq C|\psi_1 \circ G_{W^1} - \psi_2 \circ G_{W^2}|_{\mathcal{E}^q(I_j)} \leq C \varepsilon.
\end{equation}

To estimate the second term of (4.22), note that since \( d_{\mathcal{W}^1}(W^1, W^2) \leq \varepsilon \), we have \(|f_j^1 - f_j^2|_{\mathcal{E}^q(I_j)} \leq C \varepsilon\). Thus for \( r \in I_j \),

\begin{equation}
|\psi_2 \circ G_{W^2} \circ f_j^1(r) - \psi_2 \circ G_{W^2} \circ f_j^2(r)| \leq C|\psi_2|_{\mathcal{E}^p}|f_j^1(r) - f_j^2(r)|^p \leq C|\psi_2|_{\mathcal{E}^p} \varepsilon^p.
\end{equation}

Using (4.24), we write for \( r, s \in I_j \),

\begin{equation}
|f_j^1(r) - f_j^2(s)| \leq 2C|\psi_2|_{\mathcal{E}^p} \varepsilon^p.
\end{equation}

On the other hand, note that for \( k = 1, 2 \),

\begin{equation}
|\psi_2 \circ G_{W^2} \circ f_j^k(r) - \psi_2 \circ G_{W^2} \circ f_j^k(s)| \leq C|\psi_2|_{\mathcal{E}^p}|f_j^k(r) - f_j^k(s)|^p \leq C|\psi_2|_{\mathcal{E}^p}|r - s|^p,
\end{equation}

using the fact that \(|f_j^k|_{\mathcal{E}^1} \leq C\). These estimates together imply that the Hölder constant of \( \psi_2 \circ G_{W^2} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^2 \) is bounded by \( C|\psi_1|_{\mathcal{E}^p} \sup_{r, s \in I_j} \min\{|\psi_2|_{\mathcal{E}^p}|r - s|^{-\theta}, |r - s|^{p - \theta}\} \). The minimum is attained when the two bounds are equal, i.e.,

\begin{equation}
|\psi_2 \circ G_{W^2} \circ f_j^1 - \psi_2 \circ G_{W^2} \circ f_j^2|_{\mathcal{E}^q(I_j)} \leq C|\psi_2|_{\mathcal{E}^p} \varepsilon^{p - \theta}.
\end{equation}

This estimate and (4.23) prove the lemma since \(|\psi_2|_{\mathcal{E}^p(W^2)} \leq (\cos W^2)^{-1} \).

\[ \square \]

5. Proof of Theorem 2.5

The Lasota–Yorke estimate (2.7) and the compactness of the unit ball of \( \mathcal{B} \) in \( \mathcal{B}_w \) imply via the standard Hennion argument that the spectral radius of \( \mathcal{L} \) on \( \mathcal{B} \) is bounded by 1 and the essential spectral radius is bounded by \( \sigma < 1 \) (see for example [1]). Indeed, the spectral radius is 1, since if it were smaller than 1, by Lemma 3.9, we would obtain the following contradiction,

\begin{equation}
1 = m(1) = \lim_{n \to \infty} |\mathcal{L}^n m(1)| \leq C \lim_{n \to \infty} \|\mathcal{L}^n m\|_{\mathcal{B}} = 0.
\end{equation}

Our proof of Theorem 2.5 follows closely that in [18, Section 5]. Although our proofs in Sections 3 and 4 were different from those in [18] due to the countable number of singularities and the additional cutting to maintain bounded distortion, the norms are in fact similar (excluding the additional weights of \( \cos W \)) so that once the spectral gap is proved, the subsequent characterization of the peripheral spectrum of \( \mathcal{L} \) follows from the same rather general arguments.\footnote{See also [3, Appendix B] for a general strategy to prove the characterization of the peripheral spectrum once the Lasota–Yorke inequalities and several of the lemmas of Section 3.3 have been established.} We
include some of the arguments here for completeness and to point out which properties follow from our functional analytic context and which follow from previously known properties of billiards.

5.1. Peripheral Spectrum. Let \( \mathcal{V}_\theta \) be the eigenspace of \( \mathcal{L} \) associated with the eigenvalue \( e^{2\pi i \theta} \) and let \( \Pi_\theta \) be the eigenprojector onto \( \mathcal{V}_\theta \). We begin by proving the following characterization of the peripheral spectrum of \( \mathcal{L} \).

**Lemma 5.1.** Let \( \mathcal{V} = \bigoplus_\theta \mathcal{V}_\theta \). Then,

(i) \( \mathcal{L} \) restricted to \( \mathcal{V} \) has semisimple spectrum (no Jordan blocks);
(ii) \( \mathcal{V} \) consists of signed measures;
(iii) all measures in \( (iii) \) are absolutely continuous with respect to \( \bar{\mu} := \Pi_0 m \). Moreover, 1 is in the spectrum of \( \mathcal{L} \).
(iv) Let \( \mathcal{F}_{\pm n,\varepsilon} \) denote the \( \varepsilon \)-neighborhood of \( \mathcal{F}_{\pm n} \). Then for each \( \mathcal{V} \in \mathcal{N}, n \in \mathbb{N} \), we have \( \mathcal{V}(\mathcal{F}_{\pm n,\varepsilon}) \leq C_n e^\alpha \), for some constants \( C_n > 0 \).

**Proof:**

(i) Suppose there exists \( z \in \mathbb{C} \), \( |z| = 1 \), and \( h_1, h_2 \in \mathcal{B}, h_1 \neq 0 \), such that \( \mathcal{L} h_1 = zh_1 \) and \( \mathcal{L} h_2 = zh_2 + h_1 \). Then \( \mathcal{L}^n h_2 = z^n h_2 + n z^{n-1} h_1 \) so that

\[
\| \mathcal{L}^n h_2 \| \leq n \| h_1 \|_B - \| h_2 \|_B, \quad \text{for each} \ n \geq 0,
\]

which contradicts the fact that \( \| \mathcal{L}^n \| \leq \) remains bounded for all \( n \) due to \( (2.7) \).

(ii) Recall that for \( \psi \in \mathcal{C}^p(W^s) \), we have \( \psi \circ T^n \in \mathcal{C}^p(T^{-n}W^s) \). Thus by Lemma 3.9, for \( h \in \mathcal{B} \),

\[
|\mathcal{L}^n h(\psi)| = |h(\psi \circ T^n)| \leq C \| h \|_B (|\psi|_\infty + H^n_0(\psi \circ T^n))
\]

\[
\leq C \| h \|_B (|\psi|_\infty + \Lambda^{-pn} H_0^p(\psi)),
\]

where as usual, \( H_0^p(\cdot) \) is the Hölder constant with exponent \( p \) measured along curves in \( T^{-n}W^s \subseteq W^s \) and we have used \( (4.3) \).

Suppose \( \mathcal{V} \in \mathcal{V} \) with \( \mathcal{L} \mathcal{V} = z \mathcal{V} \), for some \( z \in \mathbb{C}, |z| = 1 \). Then by \( (5.2) \), for each \( n \geq 0 \),

\[
|\mathcal{V}(\psi)| = |z|^{-n} |\mathcal{L}^n \mathcal{V}(\psi)| \leq C \| \mathcal{V} \|_B (|\psi|_\infty + \Lambda^{-pn} H_0^p(\psi)).
\]

Taking the limit as \( n \to \infty \) yields \( |\mathcal{V}(\psi)| \leq C \| \mathcal{V} \|_B |\psi|_\infty \) for all \( \psi \in \mathcal{C}^p(W^s) \), so that \( \mathcal{V} \) is a measure.

(iii) By density, \( \mathcal{V}_\theta = \Pi_\theta \mathcal{C}^1(M) \). So for each \( \psi \in \mathcal{V}_\theta \), there exists \( h \in \mathcal{C}^1(M) \) such that \( \Pi_\theta h = \psi \). Now for each \( \psi \in \mathcal{C}^p(M) \),

\[
|\mathcal{V}(\psi)| = |\Pi_\theta h(\psi)| \leq \| h \|_\infty \Pi_0 1(|\psi|) = \| h \|_\infty \bar{\mu}(|\psi|).
\]

Thus \( \mathcal{V} \) is absolutely continuous with respect to \( \bar{\mu} \). Moreover, letting \( h_\nu = \frac{d\nu}{d\mu} \), we have \( h_\nu \in L^\infty(M, \bar{\mu}) \). This implies that \( \bar{\mu} \neq 0 \) since then the spectral radius of \( \mathcal{L} \) would be strictly less than 1, leading to the contradiction given by \( (5.1) \). Since \( \mathcal{L} \bar{\mu} = \bar{\mu}, \bar{\mu} \neq 0 \), then 1 belongs to the spectrum of \( \mathcal{L} \).
(iv) We give a different proof here from that in [18] due to the fact that our singularity set is countable rather than finite.

Let \( \nu \in \mathcal{S} \) and fix \( n \geq 0 \). Let \( \mathcal{S}_{-n, \varepsilon}^{\varepsilon} \) denote the \( \varepsilon \)-neighborhood of \( \mathcal{S}_{-n}^{\varepsilon} \) and let \( h_k \) be a sequence of \( \mathcal{C}^1 \) functions converging to \( \nu \) in \( \mathcal{B} \); then since \( \mathcal{L} \) is bounded, \( \mathcal{L}^n h_k \) converges to \( \mathcal{L}^n \nu \) in \( \mathcal{B} \). It is straightforward to check (applying Lemma 3.7) that \( (\mathcal{L}^n h_k)_\varepsilon (\psi) := \mathcal{L}^n h_k (1_{\mathcal{S}_{-n, \varepsilon}^{\varepsilon}} \psi) \) belongs to \( \mathcal{B}_w \) due to the uniform transversality of curves in \( \mathcal{S}_{-n}^{\varepsilon} \) to the stable cone. Then, for \( \psi \in \mathcal{C}(\mathcal{M}) \) and \( W \in \mathcal{W}^s \),

\[
\int_W (\mathcal{L}^n h_k)_\varepsilon \psi \, d\nu_W = \int_W \mathcal{L}^n h_k 1_{\mathcal{S}_{-n, \varepsilon}^{\varepsilon}} \psi \, d\nu_W = \sum_i \int_{W_i \cap T^{-n} \mathcal{S}_{-n, \varepsilon}^{\varepsilon}} h_k |DT^n|^{-1} J_{W_i} T^n \psi \circ T^n \, d\nu_W.
\]

Note that since \( W_i^n \) are created by intersections of \( W \) with \( \mathcal{S}_{-n, \varepsilon}^{\varepsilon} \), it follows that there are at most two connected components in each \( W_i^n \cap T^{-n} \mathcal{S}_{-n, \varepsilon}^{\varepsilon} \) and \( |T^n W_i^n \cap \mathcal{S}_{-n, \varepsilon}^{\varepsilon}| \leq CE \). Consequently, we estimate the above expression following (4.12),

\[
\left| \int_W (\mathcal{L}^n h_k)_\varepsilon \psi \, d\nu_W \right| \leq C \|h_k\|_1 \sum_i |W_i^n \cap T^{-n} \mathcal{S}_{-n, \varepsilon}^{\varepsilon}|^{\alpha} \frac{|T^n W_i^n|}{|W_i^n|} \leq C \varepsilon^{\alpha} \|h_k\|_1 \sum_i \frac{|T^n W_i^n|^{1-\alpha}}{|W_i^n|^{1-\alpha}} \leq C \varepsilon^{\alpha} \|h_k\|_1 C_1^n,
\]

by Lemma 3.4 with \( \zeta = 1 - \alpha \). Similarly, \( (\mathcal{L}^n h_k)_\varepsilon \) is a Cauchy sequence in \( \mathcal{B}_w \) and so must converge to \( (\mathcal{L}^n \nu)_\varepsilon (\psi) := \mathcal{L}^n \nu (1_{\mathcal{S}_{-n, \varepsilon}^{\varepsilon}} \psi) \). Then by Lemma 3.9, we have \( |\mathcal{L}^n \nu (\mathcal{S}_{-n, \varepsilon}^{\varepsilon})| \leq C_0 \|\nu\|_1 \varepsilon^\alpha \). But since \( \mathcal{L}^n \nu = z^n \nu \) for some \( z \in \mathbb{C} \), \( |z| = 1 \), we have the same bound for \( \nu (\mathcal{S}_{-n, \varepsilon}^{\varepsilon}) \). The bound for \( \nu (\mathcal{S}_{-n, \varepsilon}^{\varepsilon}) \) follows since \( T^{-n} \mathcal{S}_{-n}^{\varepsilon} = \mathcal{S}_{-n}^{\varepsilon} \) for each \( n \geq 0 \).

Since the spectrum outside the circle of radius \( \sigma < 1 \) consists of only finitely many eigenvalues of finite multiplicity and there are no Jordan blocks, the limit

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-2\pi i \theta k} \mathcal{L}^k = \Pi_{\theta}
\]

is well-defined in the uniform topology of \( L(\mathcal{B}, \mathcal{B}) \).

Additional information about the measures corresponding to the peripheral spectrum of \( \mathcal{L} \) can be proved using similar techniques as in Lemma 5.1: In other words, they are proved using properties of the Banach spaces we have defined without relying on specific properties of the billiard map. We summarize these results in our next lemma, which we state without proof since the proof can be found in [18, Lemmas 5.5 and 5.7].

Recall that an ergodic invariant probability measure \( \nu \) is said to be a physical measure if there exists a positive Lebesgue measure invariant set \( B_\nu \), with
\( \nu(B) = 1 \), such that, for each continuous function \( f \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \nu(f) \quad \forall x \in B_v.
\]

**Lemma 5.2.**  (i) There exist a finite number of \( q_i \in \mathbb{N} \) such that the spectrum of \( L \) on the unit circle is \( \bigcup_k \{ e^{2\pi i \frac{p}{q_k}} : 0 < p < q_k, p \in \mathbb{N} \} \). In addition, the set of ergodic probability measures absolutely continuous with respect to \( \overline{\mu} \) form a basis of \( \mathcal{V}_0 \).

(ii) \( T \) admits only finitely many physical probability measures and they belong to \( \mathcal{V}_0 \).

(iii) The ergodic decomposition with respect to Lebesgue and with respect to \( \overline{\mu} \) coincide. In addition, the ergodic decomposition with respect to Lebesgue corresponds to the supports of the physical measures.

The only properties of \( T \) that are used in the proof of the preceding lemma in [18] are the invertibility of \( T \) and the items in Lemma 5.1.

At this point it is useful to invoke some well-known facts about the Lorentz gas that simplify the spectral picture greatly. Recall that \( T \) has a smooth invariant measure \( d\mu = \rho dm \) where \( \rho = c \cos \phi \) and \( c \) is a normalizing constant. Since \( \rho \in C^1(M) \), we have \( \mu \in \mathcal{B} \). So by Lemma 5.1(iii), \( \mu \) is absolutely continuous with respect to \( \overline{\mu} \) and since the support of \( \mu \) is all of \( M \), it must be that \( \mu = \overline{\mu} \).

Now the ergodicity and mixing properties of \( T \) imply that the peripheral spectrum of \( L \) consists of just the simple eigenvalue at 1 with \( \mu \) as its unique normalized eigenvector. Thus the spectral projectors \( \Pi_0 \) are all zero except for \( \Pi_0 \) which can be recharacterized by \( \Pi_0 h = \lim_{n \to \infty} L^n h \). It thus follows that any probability measure \( \nu \in \mathcal{B} \) satisfies \( \Pi_0 \nu = \mu \) and this convergence occurs at an exponential rate given by the spectral radius of \( L - \Pi_0 \) on \( \mathcal{B} \). This proves item (1) of Theorem 2.5.

### 5.2. Statistical Properties.

We prove items (2) and (3) of Theorem 2.5. Given \( \phi \in C^\gamma(M), \gamma > 2\beta \) and \( \psi \in C^p(W^s) \), we define the correlation functions by

\[
C_{\phi, \psi}(n) := \mu(\phi \psi \circ T^n) - \mu(\phi) \mu(\psi).
\]

Define \( \mu_\phi = \phi \mu \). Since \( \phi \cos \phi \in C^\gamma(M) \), by Lemma 3.7 we have \( \mu_\phi \in \mathcal{B} \). Thus by Theorem 2.5(1), \( \Pi_0 \mu_\phi = \mu(\phi) \mu \) and so

\[
|\mu(\phi \psi \circ T^n) - \mu(\phi) \mu(\psi)| = |(L^n \mu_\phi - \mu(\phi) \mu)(\psi)|
\leq C \|L^n \mu_\phi - \mu(\phi) \mu\|_\mathcal{B} (|\psi|_\infty + H^p_0(\psi))
\]

and again the exponential rate of convergence is given by the spectral radius of \( L - \Pi_0 \) on \( \mathcal{B} \) as in item (1). Item (2) of Theorem 2.5 follows by noting that \( \|\mu_\phi\|_\mathcal{B} \leq C |\phi|_{C^\gamma(M)} \) by (3.24) in the proof of Lemma 3.7.
If we assume $\phi, \psi \in C^p(M)$, where $p' > \max\{p, 2\beta\}$, we can define the Fourier transform of the correlation function,$^5$

$$\hat{C}_{\phi, \psi}(z) := \sum_{n \in \mathbb{Z}} z^n C_{\phi, \psi}(n).$$

The importance of this function stems from the connection between its poles and the Ruelle resonances, which are in principal measurable in physical systems, [37, 38, 34, 35, 30].

Given the spectral picture we have established, it follows by standard arguments (see for example [18, Section 5.3]) that the function is convergent in a neighborhood of $|z| = 1$ and admits a meromorphic extension in the annulus $\{z \in \mathbb{C} : \sigma < |z| < \sigma^{-1}\}$ where $\sigma$ is from (2.7). It follows that the poles of the correlation function are in a one-to-one correspondence (including multiplicity) with the spectrum of $L$ outside the disk of radius $\sigma$. This is item (3) of Theorem 2.5.

6. PROOFS OF LIMIT THEOREMS

In this section, we show how Theorem 2.6 follows from the established spectral picture. Choose $\gamma = \max\{p, 2\beta + \epsilon\}$ for some $\epsilon > 0$. Let $g \in C^\gamma(M)$ and define $S_n g = \sum_{j=0}^{n-1} g \circ T^j$. We define the generalized transfer operator $L_g^\xi$ on $\mathcal{B}$ by, $L_g^\xi h(\psi) = h(e^{\xi} \psi \circ T)$ for all $h \in \mathcal{B}$. It is then immediate that

$$L_g^\xi h(\psi) = h(e^{S_n \xi} \psi \circ T^n), \text{ for all } n \geq 0.$$

The main element in the proofs of the limit theorems is that $L_{zg}^\xi, z \in \mathbb{C}$, is an analytic perturbation of $L = L_0$ for small $|z|$.

**Lemma 6.1.** For $g \in C^p(M)$, the map $z \mapsto L_{zg}^\xi$ is analytic for all $z \in \mathbb{C}$.

**Proof.** The lemma will follow once we show that our strong norm $\| \cdot \|_{\mathcal{B}}$ is continuous with respect to multiplication by $e^{zg}$. We will prove that for $h \in \mathcal{B}$ and $f \in C^\gamma(M), \|hf\|_{\mathcal{B}} \leq C\|f\|_{C^\gamma(M)}\|h\|_{\mathcal{B}}$ for some uniform constant $C$. Then defining the operator $\mathcal{P}_n h = L(g^n h), h \in \mathcal{B}$, the claim implies that

$$\|\mathcal{P}_n h\|_{\mathcal{B}} = \|L(g^n h)\|_{\mathcal{B}} \leq C\|L\|\|h\|_{\mathcal{B}}|g^n|_{C^\gamma(M)},$$

which allows us to conclude that the operator $\sum_{n=0}^{\infty} \frac{z^n}{n!} \mathcal{P}_n$ is well-defined on $\mathcal{B}$ and equals $L_{zg}$ since

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \mathcal{P}_n h(\psi) = h\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} g^n \cdot \psi \circ T\right) = h(e^{zg} \psi) = L_{zg} h(\psi), \text{ for } \psi \in C^p(M),$$

once we know the sum converges. We proceed to prove our claim.

---

$^5$Here we need that $\mu_\phi := \phi \mu \in \mathcal{B}$ and $\mu_\psi := \psi \mu$ belongs to the corresponding space of distributions for $T^{-1}$, which, given the invertibility of $T$, is simply $\mathcal{B}$ with the roles of $\mathcal{W}^s$ and $\mathcal{W}^u$ reversed.
By density, it suffices to prove the claim for \( h \in \mathcal{C}^1(M) \) and \( f \in \mathcal{C}^r(M) \). By Lemma 3.7, \( h f \in \mathcal{B} \). To estimate the strong stable norm of \( h f \), let \( W \in \mathcal{W}^s \) and \( \psi \in \mathcal{C}^{q}(W) \) with \( |\psi|_{W,a,q} \leq 1 \). Then

\[
\int_{W} h f \psi \, dm_{W} \leq \|h\|_{J} |W|^{a} \cos W |f|_{\mathcal{C}^{q}(W)} |\psi|_{\mathcal{C}^{q}(W)} \leq \|h\|_{J} |f|_{\mathcal{C}^{q}(W)}.
\]

Next we estimate the strong unstable norm of \( h f \). Let \( \epsilon \leq \varepsilon_0 \) and choose \( W_{1}, W_{2} \in \mathcal{W}^s \) with \( d_{\mathcal{W}^s}(W_{1}, W_{2}) < \epsilon \). For \( \ell = 1,2 \), let \( \psi_{\ell} \in W_{\ell} \) with \( |\psi_{\ell}|_{\mathcal{C}^{p}(W_{\ell})} \leq 1 \) and \( |\psi_{\ell}|_{\mathcal{C}^{q}(W_{\ell})} \leq \varepsilon \). We must estimate

\[
\int_{W_{1}} h f \psi_{1} \, dm_{W} - \int_{W_{2}} h f \psi_{2} \, dm_{W}.
\]

Recalling the notation of Section 3.1, we write \( W_{\ell} = G_{W_{\ell}}(I_{W_{\ell}}) = (r, \varphi_{W_{\ell}}(r)) : r \in I_{W_{\ell}} \) and let \( I = I_{W_{1}} \cap I_{W_{2}} \). Then,

\[
d_{q}(f \psi_{1}, f \psi_{2}) := |(f \psi_{1}) \circ G_{W_{1}} - (f \psi_{2}) \circ G_{W_{2}}|_{\mathcal{C}^{q}(I)}
\]
\[
\leq |f \circ G_{W_{1}}|_{\mathcal{C}^{q}(I)} d_{q}(\psi_{1}, \psi_{2}) + |\psi_{2} \circ G_{W_{2}}|_{\mathcal{C}^{q}(I)} |f \circ G_{W_{1}} - f \circ G_{W_{2}}|_{\mathcal{C}^{q}(I)}.
\]

The first term above is bounded by \( C|f|_{\mathcal{C}^{q}(W_{1})} \varepsilon \) by assumption on \( \psi_{1} \) and \( \psi_{2} \), where \( C \) depends only on the maximum slope in \( C^{s}(x) \). Similarly,

\[
|\psi_{2} \circ G_{W_{2}}|_{\mathcal{C}^{q}(I)} \leq C|\psi_{2}|_{\mathcal{C}^{q}(W_{2})}.
\]

For \( r \in I \), we have \( d(G_{W_{1}}(r), G_{W_{2}}(r)) \leq \epsilon \) by definition of \( d_{\mathcal{W}^s}(\cdot, \cdot) \). Thus \( |f \circ G_{W_{1}}(r) - f \circ G_{W_{2}}(r)| \leq |f|_{\mathcal{C}^{p}(M)} \epsilon^{p} \), and so by the proof of Lemma 4.3, we have \( |f \circ G_{W_{1}} - f \circ G_{W_{2}}|_{\mathcal{C}^{q}(I)} \leq |f|_{\mathcal{C}^{p}(M)} \epsilon^{p-q} \). Putting these estimates together yields

\[
d_{q}(f \psi_{1}, f \psi_{2}) \leq C|f|_{\mathcal{C}^{p}(M)} \epsilon^{p-q}.
\]

Since \( p - q \geq \beta \) and \( p > q \), we may thus estimate,

\[
\varepsilon^{-\beta} \left| \int_{W_{1}} h f \psi_{1} \, dm_{W} - \int_{W_{2}} h f \psi_{2} \, dm_{W} \right| \leq C\|h\|_{J} |f|_{\mathcal{C}^{p}(M)},
\]

which completes the estimate on the strong unstable norm and the proof of the lemma.

With the analyticity of \( z \mapsto \mathcal{L}_{z} \) established, it follows from analytic perturbation theory [25] that both the discrete spectrum and the corresponding spectral projectors of \( \mathcal{L}_{z} \) vary smoothly with \( z \). Thus, since \( \mathcal{L}_{0} \) has a spectral gap, then so does \( \mathcal{L}_{z} \) for \( z \in \mathbb{C} \) sufficiently close to 0.

**Proof of Theorem 2.6 (a).** We follow [36], making the necessary modifications to generalize to noninvariant measures \( v \in \mathcal{B} \). (See also [19].)

Let \( v \in \mathcal{B} \) be a probability measure. Assume \( |z| \) is sufficiently small so that \( \mathcal{L}_{z} \) has a spectral gap. Let \( \lambda_{z} \) be the eigenvalue of maximum modulus and denote by \( \Pi_{z} \) the associated eigenprojector. Since \( \Pi_{\lambda_{z}} v(1) = 1 \) and the spectral projectors vary continuously, we have \( \Pi_{\lambda_{z}^*} v(1) \neq 0 \) for \( z \) sufficiently small, say \( |z| < \gamma \) for some \( \gamma > 0 \). We define the moment generating function \( q(z) \) by

\[
q(z) := \lim_{n \to \infty} \frac{1}{n} \log v(e^{zS_{n}g}) = \lim_{n \to \infty} \frac{1}{n} \log \mathcal{L}_{z}^{n} v(1) = \log \lambda_{z},
\]
where the second and third equalities follows from the spectral gap of $\mathcal{L}_\nu$ and the fact that $\Pi_{\Lambda_\nu}v(1) \neq 0$. Note that $q(z)$ is independent of $\nu$ and is analytic in $|z| < \gamma$.

Let $\zeta^2$ denote the limit of the variance of $n^{-1/2} S_n g$ as $n \to \infty$ where $\{g \circ T^j\}_{j \in \mathbb{N}}$ is distributed according to the invariant measure $\mu$. (Such a $\zeta$ exists and is finite whenever the auto-correlations $C_{g,g}(k)$ are summable.) One can show as in [36, Theorem 4.3] that in fact $q'(0) = \mu(g)$, $q''(0) = \zeta^2$ and $q$ is strictly convex for real $z$ whenever $\zeta^2 > 0$.

Now let $I(u)$ be the Legendre transform of $q(z)$. Then it follows from the Gartner–Ellis Theorem [17] that for any interval $[a,b] \subset [q'(-\gamma), q'(\gamma)]$ and for any probability measure $\nu \in \mathcal{B}$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log v \left( x \in M : \frac{1}{n} S_n g(x) \in [a,b] \right) = -\inf_{u \in [a,b]} I(u),$$

which is the desired large deviation estimate with uniform rate function $I$.

**Proof of Theorem 2.6 (b).** We assume $\mu(g) = 0$ and distribute $(g \circ T^j)_{j \in \mathbb{N}}$ according to $\mu$. As above, let $\zeta^2$ denote the variance of $n^{-1/2} S_n g$ as $n \to \infty$. We consider purely imaginary $z = it$ with $|t| < \gamma$. Since $\mathcal{L}_\nu$ depends analytically on $z$, it follows from standard perturbation theory that the leading eigenvalue of $\mathcal{L}_{it \nu}$ is given by $1 - \frac{\zeta^2 t^2}{2} + O(t^3)$ for $|t| < \gamma$. It then follows using the weak dependence of $g \circ T^j$ that,

$$\lim_{n \to \infty} \mu(e^{-i \frac{\zeta}{\sqrt{n}} S_n g}) = \lim_{n \to \infty} \left( 1 - \frac{\zeta^2 t^2}{2n} \right)^n = e^{-\zeta^2 t^2/2},$$

which is the Central Limit Theorem.

The extension to noninvariant probability measures follows easily as well. Let $\nu$ be a probability measure. We still require $\mu(g) = 0$, but now distribute $(g \circ T^j)_{j \in \mathbb{N}}$ according to $\nu$. As $j$ goes to infinity, the asymptotic mean is $\nu(g \circ T^j) = \mathcal{L}^j \nu(g) \to \mu(g) = 0$ and the asymptotic variance is still $\zeta^2$ as above. At this point, there is a variety of references at our disposal, but we choose to cite [22] as a recent work since it proves both the Central Limit Theorem and the almost-sure invariance principle using spectral methods.

Since the transfer operator $\mathcal{L}_{it \nu}$ codes the characteristic function of the process $(g \circ T^j)$ in the sense of [22, Section 2.1], i.e., $\nu(e^{-it S_n g}) = \mathcal{L}_{it \nu}^n \nu(1)$, and we have proved that $\mathcal{L}_{it \nu}$ satisfies the assumptions of strong continuity in [22, Section 2.2], we may apply [22, Theorem 2.1] to conclude that $\frac{1}{\sqrt{n}} S_n g$ converges in distribution to a normal random variable with mean 0 and variance $\zeta^2$, as required.

**Proof of Theorem 2.6 (c).** The almost-sure invariance principle follows from the analytisity of the map $z \to \mathcal{L}_\nu$ and the resulting persistence of the spectral gap in a neighborhood of the origin, similar to the proofs of the previous two limit theorems. Indeed, the invariance principle holds under much weaker conditions than those present here. As in the proof of (b), noting that we have
proved our operators $\mathcal{L}_{tg}$ satisfy the assumptions of strong continuity in [22, Section 2.2], we may apply [22, Theorem 2.1] to conclude the almost-sure invariance principle in the context of the functional analytic framework we have constructed here.

\section*{Appendix A. Distortion Bounds}

The following are distortion bounds used in deriving the Lasota–Yorke estimates which hold for the Lorentz gas with both finite and infinite horizon. There exists a constant $C_d > 0$ with the following properties. Let $W^i \in \mathcal{W}^3$ and for any $n \in \mathbb{N}$, let $x, y \in W$ for some connected component $W \subset T^{-n}W^i$ such that $T^iW$ is a homogeneous stable curve for each $0 \leq i \leq n$. Then,

\begin{equation}
(A.1) \quad \left|\frac{DT^n(x)}{DT^n(y)} - 1\right| \leq C_d d_W(x, y)^{1/3} \quad \text{and} \quad \left|\frac{J_W T^n(x)}{J_W T^n(y)} - 1\right| \leq C_d d_W(x, y)^{1/3}.
\end{equation}

In particular, these bounds imply that $||DT^n||_{C^p(W)} \leq C_d ||DT^n||_{C^0(W)}$ and $|J_W T^n|_{C^p(W)} \leq C_d |J_W T^n|_{C^0(W)}$ for any $0 \leq p \leq 1/3$.

The second inequality in (A.1) is equivalent to (3.1) and is a standard distortion bound for billiards (see [7, 8] or [14] for both the finite- and infinite-horizon cases). In the proof of the distortion bound, the main idea is to prove that along a stable curve $W \in \mathcal{W}^3$, for any $x, y \in W$

\begin{equation}
(A.2) \quad \frac{d(x, y)}{\cos \varphi(x)} \leq C d(x, y)^{1/3},
\end{equation}

which follows from the definition of the homogeneity strips $H_k$ and the uniform transversality of the stable cone to horizontal lines. Indeed the first inequality of (A.1) directly follows from the estimate (A.2). More precisely, for any $x, y$ belonging to a stable curve $W$,

$$\left|\ln\frac{|DT(x)|}{|DT(y)|}\right| = \left|\ln\frac{\cos \varphi(x)}{\cos \varphi(y)} + \ln\frac{\cos \varphi(Ty)}{\cos \varphi(Tx)}\right| \leq C_1 \frac{d(x, y)}{\cos \varphi(x)} + C_2 \frac{d(Tx, Ty)}{\cos \varphi(Tx)} \leq C d(x, y)^{1/3},$$

where $C_1, C_2, C$ are positive constants and we used the hyperbolicity (2.8) in the last inequality.

By an entirely analogous argument (due to the time reversibility of the billiard map), if $W \in \mathcal{W}^u$ is an unstable curve such that $T^iW$ is a homogeneous unstable curve for $0 \leq i \leq n$, then for any $x, y \in W$,

\begin{equation}
(A.3) \quad \left|\frac{DT^n(x)}{DT^n(y)} - 1\right| \leq C_d d(T^n x, T^n y)^{1/3}.
\end{equation}

In Section 4.3, we need to compare the stable Jacobian of $T$ along different stable curves. For this, the following distortion bound is essential. Let $W^1, W^2 \in \mathcal{W}^3$, and suppose there exist $U^\ell \subset T^{-n}W^\ell$, $\ell = 1, 2$, such that $T^iU^\ell$ is a homogeneous stable curve for $0 \leq i \leq n$, and $U^1$ and $U^2$ can be put into a 1-1 correspondence by a smooth foliation $\{\gamma_x\}_{x \in U^i}$ of curves $\gamma_x \in \mathcal{W}^u$ such that $\{T^n \gamma_x\} \subset \mathcal{W}^u$ creates a 1-1 correspondence between $T^n U^1$ and $T^n U^2$. Let
\( J_{U^1} T^n \) denote the stable Jacobian of \( T^n \) along the curve \( U^\ell \). Then for \( x \in U^1 \), \( \bar{x} \in \gamma_x \cap U^2 \), we have

\[
(A.4) \quad \left| \frac{J_{U^1} T^n(x)}{J_{U^2} T^n(\bar{x})} - 1 \right| \leq C_1 d(T^n x, T^n \bar{x})^{1/3} + C_2 \theta(T^n x, T^n \bar{x}),
\]

where \( \theta(T^n x, T^n \bar{x}) \) is the angle formed by the tangent lines of \( T^n U^1 \) and \( T^n U_2 \) at \( T^n x \) and \( T^n \bar{x} \), respectively.

This distortion bound is proved as part of [14, Theorem 8.1] (see also [15, §5.8]). We explain the argument briefly, modifying the notation as necessary since the proof in [14] is written for unstable curves mapped by \( T \) while we need these bounds for stable curves mapped backwards by \( T^{-1} \). In addition, the time reversal of the context in [14] would have \( x \) and \( \bar{x} \) lying on the same unstable manifold, while in our setting, \( x \) and \( \bar{x} \) just lie on a common unstable curve. The reason these estimates hold is because we are able to choose our foliation \( \{ \gamma_x \} \) after fixing \( n \).

We relabel \( T^n U^1 = V^1 \), \( T^n U^2 = V^2 \) and \( [\omega_z]_{z \in \gamma_x} = \{ T^n \gamma_x \}_{x \in U^1} \), with the identification \( z = T^n x \). For any \( z \in V^1 \), \( \bar{z} \in \omega_z \cap V^2 \), and \( i = 0, \ldots, n \), we denote \( V_i^\ell = T^{-i} V^\ell \), \( \ell = 1, 2 \) and \( z_i = T^{-i} z \), \( \bar{z}_i = T^{-i} \bar{z} \). By [14, eq (8.6)], we have

\[
\left| \ln \frac{J_{V_i^1} T^{-1}(z_i)}{J_{V_i^2} T^{-1}(\bar{z}_i)} \right| \leq C(\rho_i^{1/2} + \theta_i + \theta_i + \rho_i), \quad i = 0, 1, \ldots, n - 1,
\]

where \( C \) is a uniform constant, \( \rho_i = |T^{-i} \omega_z| \) and \( \theta_i \) is the angle made by the tangent lines of \( V^1_i \) and \( V^2_i \) at \( z_i \), \( \bar{z}_i \), respectively.

It is important for the uniformity of this estimate that \( \omega_z = T^n(\gamma_{T^{-n} z}) \) so that \( T^{-i} \omega_z \) remains in \( W^u \) for each \( i = 0, 1, \ldots, n \). In addition, it is a consequence of [14, eq (8.9)] that

\[
\theta_i \leq C(\rho_0 i \Lambda^{-i} + \theta_0 \Lambda^{-i}).
\]

Combining this with the fact that \( \rho_i \leq C \rho_0 \Lambda^{-i} \) due to uniform hyperbolicity, we get

\[
|\ln J_{V_i^1} T^{-1}(z_i) - \ln J_{V_i^2} T^{-1}(\bar{z}_i)| \leq \text{const.} \left( \rho_0^{1/2} \Lambda^{-i/3} + \rho_0 i \Lambda^{-i} + \theta_0 \Lambda^{-i} \right).
\]

Summing over \( i = 0, \ldots, n - 1 \), we obtain (A.4) with \( x = T^{-n} z \), \( \bar{x} = T^{-n} \bar{z} \).

REFERENCES


SPECTRAL ANALYSIS OF THE TRANSFER OPERATOR FOR THE LORENTZ GAS


MARK F. DEMERS <mdemers@fairfield.edu>: Department of Mathematics and Computer Science, Fairfield University, Fairfield CT 06824, USA

HONG-KUN ZHANG <hongkun@math.umass.edu>: Department of Mathematics and Statistics, University of Massachusetts, Amherst MA 01003, USA